

# Sets of Minimal Capacity and Extremal Domains

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**ABSTRACT.** Let  $f$  be a function meromorphic in a neighborhood of infinity. The central problem in the present investigation is to find the largest domain  $D \subset \overline{\mathbb{C}}$  to which the function  $f$  can be extended in a meromorphic and single-valued manner. 'Large' means here that the complement  $\overline{\mathbb{C}} \setminus D$  is minimal with respect to (logarithmic) capacity. Such extremal domains play an important role in Padé approximation.

In the paper a unique existence theorem for extremal domains and their complementary sets of minimal capacity is proved. The topological structure of sets of minimal capacity is studied, and analytic tools for their characterization are presented; most notable are here quadratic differentials and a specific symmetry property of the Green function in the extremal domain. A local condition for the minimality of the capacity is formulated and studied. Geometric estimates for sets of minimal capacity are given.

Basic ideas are illustrated by several concrete examples, which are also used in a discussion of the principal differences between the extremality problem under investigation and some classical problems from geometric function theory that possess many similarities, which for instance is the case for Chebotarev's Problem.

## 1. Introduction

We assume that  $f$  is a function meromorphic in a neighborhood of infinity, and consider domains  $D \subset \overline{\mathbb{C}}$  to which the function  $f$  can be extended in a meromorphic and single-valued manner. The basic problem of our investigation is to find the domain with a complement of minimal (logarithmic) capacity. It will be shown that for any function  $f$  that is meromorphic at infinity such a domain exists and is essentially unique. The domain is called extremal, and its complement is called the minimal set (or the set of minimal capacity). Formal definitions are given in the Sections 2 and 3.

Extremal domains play an important role in rational approximation, and there especially in the convergence theory of Padé approximants (cf. [6], [19], [21], [20], [31], [32], [33], [34], [36], [38], [2] Chapter 6). Variants of the concept will also be useful in other areas of rational approximation and the theory of orthogonal polynomials.

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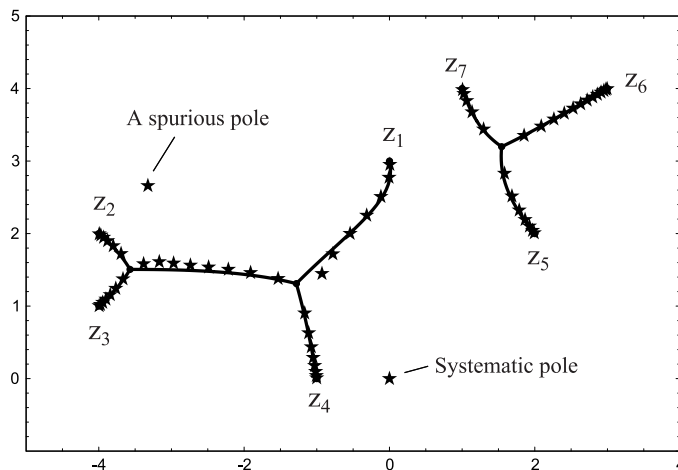


FIGURE 1. The poles of the Padé approximant  $[63/62]_f$  to the function (1.1) are represented by stars, and the associated minimal set  $K_0(f, \infty)$  is represented by 8 unbroken lines.

Several elements of the material in the present article have already been studied in [28], [29], and [30]. Results from there will be revisited, proofs will be redone, and the whole concept will be extended and reformulated.

**1.1. A Concrete Example.** As an illustration of the role played by extremal domains in the theory of Padé approximation, we consider a concrete example. Let  $f$  be the algebraic function defined by

$$f(z) := \sqrt[4]{\prod_{j=1}^4 (1 - z_j/z)} + \sqrt[3]{\prod_{j=5}^7 (1 - z_j/z)} \quad (1.1)$$

with 7 branch points  $z_1, \dots, z_7$  that have been chosen rather arbitrarily, but with the intention to get an evenly spread out configuration. The seven values are given in (6.19), further below, but their location can readily be read from Figure 1.

The rather simple construction of the function  $f$  makes it easy to understand all possible meromorphic and single-valued continuations of  $f$ . Indeed,  $f$  possesses a single-valued continuation throughout a domain  $D \subset \overline{\mathbb{C}}$  if, and only if,  $\infty \in D$  and if each of the two sets  $\{z_1, \dots, z_4\}$  and  $\{z_5, z_6, z_7\}$  of branch points is connected in the complement  $\overline{\mathbb{C}} \setminus D$ .

The union of the 8 arcs in Figure 1 form the set of minimal capacity for the function  $f$ , which we denote by  $K_0(f, \infty)$ , and by  $D_0(f, \infty) := \overline{\mathbb{C}} \setminus K_0(f, \infty)$  we denote the extremal domain. Their definition and details about the calculation of the minimal set  $K_0(f, \infty)$  will be given in Section 2 and in the discussion of Example 6.5 in Section 6, further below.

Let  $[63/62]_f$  be the Padé approximant of numerator and denominator degree 63 and 62, respectively, to the function  $f$  developed at infinity. In Figure 1, the poles of this approximant are represented by stars. For any  $n \in \mathbb{N}$  the Padé approximant  $[n+1/n]_f = p/q$  is defined by the relation

$$f(z)q\left(\frac{1}{z}\right) - p\left(\frac{1}{z}\right) = O(z^{-2n-2}) \quad \text{as } z \rightarrow \infty \quad (1.2)$$

with  $p$  and  $q$  polynomials of degree at most  $n + 1$  and  $n$ , respectively. An comprehensive introduction to Padé approximation can be found in [2].

The connection between Padé approximation and the minimal set  $K_0(f, \infty)$  will be established in the next theorem, which covers functions of type (1.1). It has been proved in [38] (cf. also [2] Theorem 6.6.9), and is given here in a somewhat shortened and specialized form.

**THEOREM 1.** *For  $n \rightarrow \infty$ , the Padé approximants  $[n + 1/n]_f$  converge to the function (1.1) in capacity in the extremal domain  $D_0(f, \infty) \subset \overline{\mathbb{C}}$  associated with  $f$ , and this convergence is optimal in the sense that it does not hold throughout any domain  $\tilde{D} \subset \overline{\mathbb{C}}$  with  $\text{cap}(\tilde{D} \setminus D_0(f, \infty)) > 0$ .*

Theorem 1 shows that extremal domains are convergence domains for Padé approximants, and this is also the case for our concrete example. In Figure 1 we observe that 61 out of 63 poles of the Padé approximant  $[63/62]_f$  are distributed very nicely along the 8 arcs that form the minimal set  $K_0(f, \infty) = \overline{\mathbb{C}} \setminus D_0(f, \infty)$ . They are asymptotically distributed in accordance to the equilibrium distribution on the minimal set  $K_0(f, \infty)$  (cf. [38], Theorem 1.8), and they mark the places, where we don't have convergence.

There are two poles that step out of line, and each one by a different reason: One of them lies close to the origin, where it approximates the simple pole of the function  $f$  at the origin. Because of its correspondence to a pole of  $f$ , it is called systematic.

The other one, which lies at  $z = -3.35 + 2.66i$ , does not correspond to a singularity of the function  $f$ , and does obviously also not belong to any of the chains of poles along the arcs in  $K_0(f, \infty)$ . Such poles are called spurious in the theory of Padé approximation. Spurious poles always appear in combination with a nearby zero of the approximant. These pairs of poles and zeros are close to cancellation. They are a phenomenon that unfortunately cannot be ignored in Padé approximation (cf. [39], [37], or [2] Chapter 6). Convergence in capacity is compatible with the possibility of such spurious poles.

The convergence in capacity in Theorem 1 implies that almost all poles of the Padé approximants  $[n + 1/n]_f$  have to leave the extremal domain  $D_0(f, \infty)$ ; they cluster on the minimal set  $K_0(f, \infty)$ . That they do this in a rather regular way is shown in Figure 1. The picture does not change much for other values of  $n$  only that the location, and possibly also the number of spurious poles may be different in each case.

If one wants to summarize the somewhat complicated convergence theory for diagonal Padé approximants in a short sentence one can say that extremal domains are for Padé approximants what discs are for power series.

**1.2. The Outline of the Manuscript.** In the next two Sections 2 and 3, two alternative formal definitions are given for the extremality problem under investigation. In the second approach, the role of the function  $f$  is taken over by a concrete Riemann surface  $\mathcal{R}$  over  $\overline{\mathbb{C}}$ . Both formulations are equivalent.

Illustrative examples are discussed in Section 6, but before that in the two Sections 4 and 5, general results about minimal sets and extremal domains are formulated and discussed. All proofs are postponed to later sections.

In Section 7, a local version of the extremality problem is formulated and discussed. After that in Section 8, the extremality problem is compared with some

classical problems from geometric functions theory. For such problems there exists a broad range of tools and techniques, as for instance, boundary and inner variational methods, methods of extremal length, and techniques connected with quadratic differentials (cf. [24], [5], [13], [14]). Some of these ideas will play a role in our investigation. We shall use a solution of one of these problems as building block in one of our proofs.

Practically, no proofs are given in the Sections 2 - 5 and 7; they all are all postponed to the Sections 9 and 10. In Section 11, several auxiliary results from potential theory and geometric function theory are assembled, of which some have been modified quite substantially in order to fit their purpose in the present paper.

**1.3. Some Special Aspects.** It is a typical feature of the approach chosen in the present article that a general existence and uniqueness proof is put at the beginning of the analysis. This strategy has the advantage of giving great methodological liberty in later proofs of special properties. At these later stages, the knowledge of unique existence offers a free choice between different methods and techniques from the tool boxes of geometric function theory; and because of the uniqueness it is always clear that one is dealing with the same well defined object. The prize to be paid for this strategy is a rather abstract and somewhat heavy machinery for the uniqueness proof. The main tools there are potential-theoretic in nature.

It has been mentioned, and hopefully also illustrated by the introductory example (1.1), that extremality with respect to the logarithmic capacity arises in a very natural way in connection with diagonal Padé approximants. In rational approximation also other types of capacity are of interest, as for instance, condenser capacity or capacities in external fields, which become relevant in connection with rational interpolants (cf. [35]) or with essentially non-diagonal Padé approximants. The specific form of tools and methods in the present analysis should be helpful for such potential generalizations.

## 2. Basic Definitions and Unique Existence

In the present section we introduce basic definitions and formulate a theorem about the unique existence of a solution of the extremality problem.

Throughout the whole paper, we assume that  $f$  is a function meromorphic in a neighborhood of infinity, and denote its meromorphic extensions by the same symbol  $f$ . By  $\text{cap}(\cdot)$  we denote the (logarithmic) capacity.

### 2.1. The Definition of Problem $(f, \infty)$ .

**DEFINITION 1.** A domain  $D \subset \overline{\mathbb{C}}$  is called *admissible* for Problem  $(f, \infty)$  if

- (i)  $\infty \in D$ , and if
- (ii)  $f$  has a single-valued meromorphic continuation throughout  $D$ .

By  $\mathcal{D}(f, \infty)$  we denote the set of all admissible domains  $D$  for Problem  $(f, \infty)$ . A compact set  $K \subset \mathbb{C}$  is called *admissible* for Problem  $(f, \infty)$  if it is the complement  $\overline{\mathbb{C}} \setminus D$  of an admissible domain  $D \in \mathcal{D}(f, \infty)$ . By  $\mathcal{K}(f, \infty)$  we denote the set of all admissible compact sets  $K$  for Problem  $(f, \infty)$ .

Instead of meromorphic continuations, one could also consider analytic continuations in condition (ii) of Definition 1 without essentially changing the whole concept. This later option has been taken in [28], [29], and [30]. Meromorphic continuations have been chosen here because of their natural affiliation with rational approximation.

**DEFINITION 2.** *A compact set  $K_0 = K_0(f, \infty) \subset \mathbb{C}$  is called minimal (or more lengthy: a set of minimal capacity with respect to Problem  $(f, \infty)$ ) if the following three conditions are satisfied:*

- (i)  $K_0 \in \mathcal{K}(f, \infty)$ .
- (ii) We have

$$\text{cap}(K_0) = \inf_{K \in \mathcal{K}(f, \infty)} \text{cap}(K). \quad (2.1)$$

- (iii) We have  $K_0 \subset K_1$  for all  $K_1 \in \mathcal{K}(f, \infty)$  that satisfy condition (ii) with  $K_0$  replaced by  $K_1$ .

The domain  $D_0(f, \infty) := \overline{\mathbb{C}} \setminus K_0(f, \infty)$  is called extremal with respect to Problem  $(f, \infty)$  (or short: extremal domain).

By  $\mathcal{K}_0(f, \infty)$  we denote the set of all admissible compact sets  $K$  of minimal capacity, i.e., all sets  $K \in \mathcal{K}(f, \infty)$  that satisfy condition (ii), but not necessarily condition (iii), and by  $\mathcal{D}_0(f, \infty)$  the set of all admissible domains  $D \in \mathcal{D}(f, \infty)$  such that  $\overline{\mathbb{C}} \setminus D \in \mathcal{K}_0(f, \infty)$ .

With the introduction of the set of admissible domains  $\mathcal{D}(f, \infty)$  and the definition of the extremal domain  $D_0(f, \infty)$  together with its complementary minimal set  $K_0(f, \infty)$ , Problem  $(f, \infty)$  is fully defined. The problem depends solely on the function  $f$  given in neighborhood of infinity.

The point infinity plays a very special role for the function  $f$  and also in the definition of the (logarithmic) capacity, which is reflected in condition (i) of Definition 1. This special role is the reason why the symbol  $\infty$  has been used besides of  $f$  for the designation of Problem  $(f, \infty)$ .

**2.2. Unique Existence.** One of the central results in the present paper is the following existence and uniqueness theorem.

**THEOREM 2 (Unique Existence Theorem).** *For any function  $f$ , which is meromorphic in a neighborhood of infinity, there uniquely exists a minimal set  $K_0(f, \infty)$  and correspondingly a unique extremal Domain  $D_0(f, \infty)$  with respect to Problem  $(f, \infty)$ .*

Among the three conditions in Definition 2, condition (ii) is most important, and condition (iii) plays only an auxiliary role. The situation becomes evident by the next proposition.

**PROPOSITION 1.** *Elements of the set  $\mathcal{K}_0(f, \infty)$  differ at most in a set of capacity zero, and we have*

$$K_0(f, \infty) = \bigcap_{K \in \mathcal{K}_0(f, \infty)} K. \quad (2.2)$$

The concept of extremal domains is most interesting if the function  $f$  has branch points. In the absence of branch points, the concept becomes in a certain sense trivial, as the next proposition shows.

PROPOSITION 2. *If the function  $f$  of Problem  $(f, \infty)$  possesses no branch points, then the extremal domain  $D_0(f, \infty)$  coincides with the Weierstrass domain  $W_f \subset \overline{\mathbb{C}}$  for meromorphic continuation of the function  $f$  starting at  $\infty$ .*

In Section 6 we shall discuss several concrete examples of functions  $f$  together with their extremal domains  $D_0(f, \infty)$  and minimal sets  $K_0(f, \infty)$ . These examples should give more substance to the formal definitions in the present section.

Several classical extremality problems from geometric function theory that are defined by purely geometric constraints are reviewed in Section 8. There exist similarities with Problem  $(f, \infty)$ , but there are also essential differences. The intention of the selection of examples in Section 6 has been to illustrate these differences.

### 3. An Alternative Definition

In the present section a definition of the extremality problem is given that is equivalent to Problem  $(f, \infty)$ , but the role of the function  $f$  is taken over by a Riemann surface  $\mathcal{R}$ . Of course, single-valuedness, or better its absence, lies at the heart of the idea of a Riemann surface, and so the alternative approach may shed light on the geometric background of Problem  $(f, \infty)$ . Since in all later sections, with the only exception of Subsection 4.2, only Problem  $(f, \infty)$  will be used as reference point, the alternative definition in the present section can be skipped in a first reading.

Let  $\mathcal{R}$  be a Riemann surface over  $\overline{\mathbb{C}}$ , not necessarily unbounded, and let  $\pi : \mathcal{R} \rightarrow \overline{\mathbb{C}}$  be its canonical projection. We assume that  $\infty \in \pi(\mathcal{R})$ .

#### 3.1. The Definition of Problem $(\mathcal{R}, \infty^{(0)})$ .

DEFINITION 3. *Let  $\infty^{(0)}$  be a point on the Riemann surface  $\mathcal{R}$  with  $\pi(\infty^{(0)}) = \infty$ . Then a domain  $D \subset \mathcal{R}$  is called admissible for Problem  $(\mathcal{R}, \infty^{(0)})$  if the following two conditions are satisfied:*

- (i)  $\infty^{(0)} \in D$ .
- (ii) *The domain  $D$  is planar (also called schlicht), i.e.,  $\pi|_D$  is univalent, or in other words, we have  $\text{card}((\pi^{-1} \circ \pi)(\{\zeta\}) \cap D) = 1$  for all  $\zeta \in D$ .*

*By  $\mathcal{D}(\mathcal{R}, \infty^{(0)})$  we denote the set of all admissible domains  $D \subset \mathcal{R}$  for Problem  $(\mathcal{R}, \infty^{(0)})$ .*

DEFINITION 4. *A compact set  $K \subset \mathbb{C}$  is admissible for Problem  $(\mathcal{R}, \infty^{(0)})$  if it is of the form  $K := \overline{\mathbb{C}} \setminus \pi(D)$  with  $D \in \mathcal{D}(\mathcal{R}, \infty^{(0)})$ .*

*By  $\mathcal{K}(\mathcal{R}, \infty^{(0)})$  we denote the set of all admissible compact sets  $K \subset \mathbb{C}$  for Problem  $(\mathcal{R}, \infty^{(0)})$ .*

Notice that in contrast to admissible domains  $D \in \mathcal{D}(f, \infty)$ , now admissible domains  $D \in \mathcal{D}(\mathcal{R}, \infty^{(0)})$  are subdomains of the Riemann surface  $\mathcal{R}$ , while the admissible compact sets  $K \in \mathcal{K}(\mathcal{R}, \infty^{(0)})$  remain to be subsets of  $\mathbb{C}$  like it has been the case in Definition 1.

Analogously to Definition 2, we define the minimal set and the extremal domain for Problem  $(\mathcal{R}, \infty^{(0)})$  as follows.

DEFINITION 5. *A compact set  $K_0 = K_0(\mathcal{R}, \infty^{(0)}) \subset \overline{\mathbb{C}}$  is called minimal with respect to Problem  $(\mathcal{R}, \infty^{(0)})$  if the following three conditions are satisfied:*

- (i)  $K_0 \in \mathcal{K}(\mathcal{R}, \infty^{(0)})$ .
- (ii) We have

$$\text{cap}(K_0) = \inf_{K \in \mathcal{K}(\mathcal{R}, \infty^{(0)})} \text{cap}(K) \quad (3.1)$$

$$= \inf_{D \in \mathcal{D}(\mathcal{R}, \infty^{(0)})} \text{cap}(\overline{\mathbb{C}} \setminus \pi(D)). \quad (3.2)$$

- (iii) We have  $K_0 \subset K_1$  for all  $K_1 \in \mathcal{K}(\mathcal{R}, \infty^{(0)})$  that satisfy assertion (ii) with  $K_0$  replaced by  $K_1$ .

A domain  $D_0 \in \mathcal{D}(\mathcal{R}, \infty^{(0)})$  that satisfies  $\overline{\mathbb{C}} \setminus \pi(D_0) = K_0(\mathcal{R}, \infty^{(0)})$  is called extremal with respect to Problem  $(\mathcal{R}, \infty^{(0)})$ , and it is denoted by  $D_0(\mathcal{R}, \infty^{(0)})$ .

By  $\mathcal{K}_0(\mathcal{R}, \infty^{(0)})$  we denote the set of all compact sets  $K$  that satisfy the two conditions (i) and (ii), but not necessarily condition (iii).

### 3.2. Unique Existence and Equivalence.

For any Riemann surface  $\mathcal{R}$  over  $\overline{\mathbb{C}}$ , there exists a meromorphic function  $f$  such that  $\mathcal{R} = \mathcal{R}_f$  is the natural domain of definition of  $f$ . On the other hand, the meromorphic continuation of a given function  $f$ , which is meromorphic in a neighborhood of infinity, defines a Riemann surface  $\mathcal{R}_f$  over  $\overline{\mathbb{C}}$  that contains a point  $\infty^{(0)} \in \mathcal{R}_f$  with  $\pi(\infty^{(0)}) = \infty$ , and this surface  $\mathcal{R}_f$  is the natural domain of definition for the function  $f$ . From these observations we can conclude that the two Problems  $(f, \infty)$  and  $(\mathcal{R}_f, \infty^{(0)})$  are equivalent.

It is an immediate consequence of the equivalence of both problems that the existence and uniqueness of a solution to Problem  $(f, \infty)$  formulated in Theorem 2 carries over to Problem  $(\mathcal{R}, \infty^{(0)})$ . Details are formulated in the next theorem.

**THEOREM 3.** (i) For any Riemann surface  $\mathcal{R}$  over  $\overline{\mathbb{C}}$  with  $\infty^{(0)} \in \mathcal{R}$  and  $\pi(\infty^{(0)}) = \infty$ , there uniquely exists a minimal set  $K_0 = K_0(\mathcal{R}, \infty^{(0)}) \subset \mathbb{C}$  for Problem  $(\mathcal{R}, \infty^{(0)})$ , and correspondingly, there also uniquely exists an extremal domain  $D_0 = D_0(\mathcal{R}, \infty^{(0)}) \subset \mathcal{R}$ .

(ii) Let the Riemann surface  $\mathcal{R} = \mathcal{R}_f$  be the natural domain of definition for the function  $f$ , and let  $f$  be assumed to be meromorphic in a neighborhood of infinity. Then the two extremal domains  $D_0(f, \infty)$  and  $D_0(\mathcal{R}_f, \infty^{(0)})$  of the Definitions 2 and 5, respectively, are identical up to the canonical projection  $\pi : \mathcal{R}_f \rightarrow \overline{\mathbb{C}}$ , i.e., we have

$$D_0(f, \infty) = \pi \left( D_0(\mathcal{R}_f, \infty^{(0)}) \right). \quad (3.3)$$

Further, we have

$$K_0(f, \infty) = K_0(\mathcal{R}_f, \infty^{(0)}). \quad (3.4)$$

**PROOF.** We assume that the function  $f$  has the Riemann surface  $\mathcal{R} = \mathcal{R}_f$  as its natural domain of definition and that the function element of  $f$  at the point  $\infty^{(0)} \in \mathcal{R}_f$ ,  $\pi(\infty^{(0)}) = \infty$ , is identical with the function  $f$  at  $\infty \in \overline{\mathbb{C}}$ .

It immediately follows from the two Definitions 1 and 3 that for each domain  $\tilde{D} \in \mathcal{D}(\mathcal{R}_f, \infty^{(0)})$  we have  $\pi(\tilde{D}) \in \mathcal{D}(f, \infty)$ , and conversely, for each domain  $D \in \mathcal{D}(f, \infty)$  there exists an admissible domain  $\tilde{D} \in \mathcal{D}(\mathcal{R}_f, \infty^{(0)})$  with  $\pi(\tilde{D}) = D$ .

After these preparations, the theorem is an immediate consequence of the correspondence between the two sets  $\mathcal{D}(\mathcal{R}_f, \infty^{(0)})$  and  $\mathcal{D}(f, \infty)$  together with the two Definitions 2, 5, and Theorem 2.  $\square$

The equivalence of the two Problems  $(f, \infty)$  and  $(\mathcal{R}_f, \infty^{(0)})$  allows us to opt freely for one of the two approaches. In the present investigation we carry out the analysis in the framework of Problem  $(f, \infty)$ . However, in applications it is sometimes favorable to start from a Riemann surface  $\mathcal{R}$ . This approach will also give the intuitive background for the discussion of concrete examples in Section 6.

#### 4. Topological Properties

Extremal problems in geometric function theory often lead to topologically simply structured and smooth solutions. In the next two sections it will be shown that a similar situation can be observed in our present investigations.

In Subsection 4.1 we address topological properties of the minimal set  $K_0(f, \infty)$ , and corresponding results for the minimal set  $K_0(\mathcal{R}, \infty^{(0)})$  associated with Problem  $(\mathcal{R}, \infty^{(0)})$  are given in Subsection 4.2.

**4.1. Topological Properties of the Set  $K_0(f, \infty)$ .** The main result in the present section is a structure theorem for the minimal set  $K_0(f, \infty)$ . As usual, the function  $f$  is assumed to be meromorphic in a neighborhood of infinity.

**THEOREM 4 (Structure Theorem).** *Let the function  $f$  be meromorphic in a neighborhood of infinity, and let  $K_0 = K_0(f, \infty)$  be the minimal set for Problem  $(f, \infty)$ . There exist two sets  $E_0, E_1 \subset \mathbb{C}$  and a family  $\{J_j\}_{j \in I}$  of open and analytic Jordan arcs such that*

$$K_0(f, \infty) = E_0 \cup E_1 \cup \bigcup_{j \in I} J_j, \quad (4.1)$$

and the components in (4.1) have the following properties:

- (i) *We have  $\partial E_0 \subset \partial D_0(f, \infty)$ , and at each point  $z \in \partial E_0$  the meromorphic continuation of the function  $f$  has a non-polar singularity for at least one approach out of  $D_0 = D_0(f, \infty)$ . The set  $E_0 \subset K_0$  is compact and polynomial-convex, i.e.,  $\overline{\mathbb{C}} \setminus E_0$  is connected.*
- (ii) *At each point  $z \in E_1$  the function  $f$  has meromorphic continuations out of  $D_0$  from all possible sides, and these continuations lead to more than 2 different function elements at the point  $z$ . The set  $E_1$  is discrete in  $\overline{\mathbb{C}} \setminus E_0$ .*
- (iii) *All Jordan arcs  $J_j$ ,  $j \in I$ , are contained in  $\overline{\mathbb{C}} \setminus (E_0 \cup E_1)$ , they are pairwise disjoint, the function  $f$  has meromorphic continuations to each point  $z \in J_j$ ,  $j \in I$ , from both sides of  $J_j$  out of  $D_0$ , and these continuations lead to 2 different function elements at each point  $z \in J_j$ ,  $j \in I$ .*

*The properties (i), (ii), and (iii) fully characterize all components on the right-hand side of (4.1).*

**REMARK 1.** *The family of Jordan arcs  $\{J_j\}_{j \in I}$  and also the set  $E_1$  in (4.1) is empty if, and only if, all possible meromorphic continuations of the function  $f$  are single-valued, i.e., if the function  $f$  has no branch points. This situation has already been addressed in Proposition 2.*

It follows from Theorem 4 that the boundary  $\partial D_0(f, \infty)$  is smooth everywhere on  $\partial D_0(f, \infty) \setminus (\partial E_0 \cup E_1)$ . More information about this aspect is given in the next theorem.



**THEOREM 5.** *The set  $K_0(f, \infty) \setminus E_0$  is locally connected, and only a finite number ( $> 2$ ) of arcs  $J_j$ ,  $j \in I$ , meets at each point of the set  $E_1$ .*

In the next section (cf. Remark 2), we shall see that the arcs  $J_j$  that meet at a point  $z \in E_1$  form a regular star at  $z$ .

Before we close the present subsection, we will discuss the two influences that determine the structure of the minimal set  $K_0(f, \infty)$  in an informal way.

The principle of minimal capacity of the set  $K_0(f, \infty)$  implies that the extremal domain  $D_0(f, \infty)$  is as large as possible, and consequently it extends up to the natural boundary of the function  $f$  (see also Definition 8 in Subsection 7.1, further below). On the other hand, the requirement of single-valuedness of the function  $f$  in  $D_0(f, \infty)$  can in general only be avoided by cuts in the complex plane  $\mathbb{C}$ ; these cuts separate different branches of the function  $f$ .

Both aspects, maximal extension and the principle of single-valuedness, find a specific balance in the topological structure of the minimal set  $K_0(f, \infty)$ . On one hand, there is the compact subset  $E_0 \subset K_0(f, \infty)$ , where on  $\partial E_0$  meromorphic extensions of the function  $f$  find a natural boundary. On the other hand, there is the part  $K_0(f, \infty) \setminus E_0$  of  $K_0(f, \infty)$ , which essentially consists of analytic Jordan arcs  $J_j$ ,  $j \in I$ , which cut  $\mathbb{C} \setminus E_0$  in such a way that different branches of the function  $f$  are separated. They can be chosen with much liberty, and therefore optimization is possible. This optimization is done according to the principle of minimal capacity. We shall see in Section 5, and more specifically in Section 7, how a balance between forces leads to a state of equilibrium that determines the Jordan arcs  $J_j$ ,  $j \in I$ .

**4.2. Topological Properties of the Set  $K_0(\mathcal{R}, \infty^{(0)})$ .** From Theorem 3 we know that the two Problems  $(f, \infty)$  and  $(\mathcal{R}, \infty^{(0)})$  have equivalent solutions if there exists an appropriate relationship between the Riemann surface  $\mathcal{R}$  and the function  $f$ . As a consequence of this equivalence, there exists a description of the topological properties of the set  $K_0(\mathcal{R}, \infty^{(0)})$  that corresponds to that given in Theorem 4. However, now the function  $f$  is no longer available, and its role has to be taken over by properties of the Riemann surface  $\mathcal{R}$ .

Let  $\mathcal{R}$  be a Riemann surface over  $\overline{\mathbb{C}}$ . By  $\partial D$  and  $\overline{D}$  we denote the boundary and the closure of a domain  $D \subset \mathcal{R}$  in  $\mathcal{R}$ . Further, we denote the set of all branch points of  $\mathcal{R}$  by  $Br(\mathcal{R}) \subset \mathcal{R}$ , and the relative boundary of the Riemann surface  $\mathcal{R}$  over  $\overline{\mathbb{C}}$  by  $\partial \mathcal{R}$ . We set  $\tilde{\mathcal{R}} := \mathcal{R} \cup \partial \mathcal{R}$ . If the Riemann surface  $\mathcal{R}$  is compact, then we have  $\partial \mathcal{R} = \emptyset$ .

The canonical projection  $\pi : \mathcal{R} \rightarrow \overline{\mathbb{C}}$  can be extended continuously to a projection  $\tilde{\pi} : \tilde{\mathcal{R}} \rightarrow \overline{\mathbb{C}}$ . We continue to denote the boundary of a domain  $D$  in  $\tilde{\mathcal{R}}$  by the same symbol  $\partial D$  as has been done in  $\mathcal{R}$ .

After this preparations, we are ready to formulate the analog of Theorem 4 for Problem  $(\mathcal{R}, \infty^{(0)})$ .

**THEOREM 6.** *Let  $D_0 = D_0(\mathcal{R}, \infty^{(0)}) \subset \mathcal{R}$  and  $K_0 = K_0(\mathcal{R}, \infty^{(0)}) \subset \overline{\mathbb{C}}$  be the uniquely existing extremal domain and minimal set, respectively, for Problem  $(\mathcal{R}, \infty^{(0)})$ . Like in Theorem 4, there exist two sets  $E_0, E_1 \subset \mathbb{C}$  and a family  $\{J_j\}_{j \in I}$  of analytic, open Jordan arcs in  $\mathbb{C}$  such that representation (4.1) holds true with*

$K_0(f, \infty)$  replaced by  $K_0(\mathcal{R}, \infty^{(0)})$ , i.e., we have

$$K_0(\mathcal{R}, \infty^{(0)}) = E_0 \cup E_1 \cup \bigcup_{j \in I} J_j. \quad (4.2)$$

In the new situation, the components  $E_0, E_1$ , and  $\{J_j\}_{j \in I}$  in (4.2) can be characterized by the following properties:

- (i) The boundary  $\partial E_0$  of the compact set  $E_0 \subset K_0$  is equal to

$$\tilde{\pi}((\partial D_0 \cap \partial \mathcal{R}) \cup (Br(\mathcal{R}) \cap \overline{D_0})), \quad (4.3)$$

and the set  $E_0$  is the polynomial-convex hull of  $\partial E_0$ . (For a definition, see Definition 22 in Subsection 11.1, further below).

- (ii) The set  $E_1 \subset K_0$  is equal to

$$E_1 := \{ z \in K_0 \setminus E_0 \mid \text{card}(\pi^{-1}(\{z\}) \cap \partial D_0) > 2 \} \quad (4.4)$$

with  $\pi$  being the canonical projection of  $\mathcal{R}$  and not that of  $\tilde{\mathcal{R}}$ . The set  $E_1 \subset K_0$  is discrete in  $\overline{\mathcal{C}} \setminus E_0$ .

- (iii) If  $I \neq \emptyset$ , then  $K_0 \setminus (E_0 \cup E_1)$  is the disjoint union of the analytic Jordan arcs  $J_j$ ,  $j \in I$ . For each point  $z \in J_j$ ,  $j \in I$ , we have

$$\text{card}(\pi^{-1}(\{z\}) \cap \partial D_0) = 2. \quad (4.5)$$

## 5. Analytic Characterizations

We now come to analytic characterizations of the Jordan arcs  $J_j$ ,  $j \in I$ , in the minimal set  $K_0(f, \infty)$  for Problem  $(f, \infty)$ . One method is based on quadratic differentials, and a related one involves the  $S$ -property (symmetry-property) of the extremal domain  $D_0(f, \infty)$ . In the last subsection we consider the special case that the set  $E_0$  in Theorem 4 is finite, which leads to the interesting special case of rational quadratic differentials.

All results in the present section are formulated in the framework of Problem  $(f, \infty)$ . Their transfer to Problem  $(\mathcal{R}, \infty^{(0)})$  is easily possible with the tools presented in Section 3 and Subsection 4.2.

**5.1. The  $S$ -Property.** A characteristic property of the extremal domain  $D_0 = D_0(f, \infty)$  for Problem  $(f, \infty)$  is a specific behavior of the Green function  $g_{D_0}(\cdot, \infty)$  on the Jordan arcs  $J_j$ ,  $j \in I$ , in  $K_0(f, \infty)$  that have been introduced in (4.1) of Theorem 4. For a definition of the Green function we refer to Subsection 11.3, further below.

**THEOREM 7.** *Under the assumptions made in Theorem 4, we have*

$$\frac{\partial}{\partial n_+} g_{D_0}(z, \infty) = \frac{\partial}{\partial n_-} g_{D_0}(z, \infty) \quad \text{for all } z \in J_j, j \in I, \quad (5.1)$$

with  $\partial/\partial n_+$  and  $\partial/\partial n_-$  denoting the normal derivatives to both sides of the arcs  $J_j$ ,  $j \in I$ , that have been introduced in (4.1) of Theorem 4.

The symmetric boundary behavior (5.1) of the Green function  $g_{D_0}(\cdot, \infty)$  is called the  $S$ -property of the extremal domain  $D_0(f, \infty)$ . In Section 7, below, it will be shown that the  $S$ -property can be interpreted as a local condition for the minimality (2.1) in Definition 2.

While in Theorem 7 we get the  $S$ -property as a consequence of the minimality (2.1) in Definition 2, it will be proved in Theorem 11 in Subsection 7.3 that the

$S$ -property is even equivalent to the minimality (2.1). As a consequence of this further going result it follows that the  $S$ -property can also be used as an alternative characterization of the extremal domain  $D_0(f, \infty)$ .

Notice that  $I \neq \emptyset$  in (5.1) implies  $\text{cap}(K_0) > 0$ , and consequently, in this case, the Green function  $g_{D_0}(\cdot, \infty)$  in (5.1) exists in a proper sense (cf. Subsection 11.3, further below). If on the other hand, we have  $I = \emptyset$ , then relation (5.1) is void.

From Theorem 4 we now know that the arcs  $J_j$ ,  $j \in I$ , are analytic. Hence, the Green function  $g_{D_0}(\cdot, \infty)$  has harmonic continuations across each arc  $J_j$  from both sides (cf. Subsection 11.3), and consequently the normal derivatives in (5.1) exist for each  $z \in J_j$ ,  $j \in I$ .

**5.2. Quadratic Differentials.** The  $S$ -property can be described in an equivalent way by quadratic differentials. We say that a smooth arc  $\gamma$  with parametrization  $z : [0, 1] \rightarrow \overline{\mathbb{C}}$  is a trajectory of the quadratic differential  $q(z)dz^2$  if we have

$$q(z(t))\dot{z}(t)^2 < 0 \quad \text{for all } t \in (0, 1). \quad (5.2)$$

We note that there exists an associated family of orthogonal trajectories, which are defined by the same relation (5.2), but with an inequality showing in the other direction. As general reference to quadratic differentials and their trajectories we use [40] or [10]. Some of its local properties are assembled in Subsection 11.5, further below.

**THEOREM 8.** *Let  $D_0 = D_0(f, \infty)$ ,  $E_0, E_1 \subset \overline{\mathbb{C}}$ , and  $\{J_j\}_{j \in I}$  be the objects introduced in Theorem 4, and let  $g_{D_0}(\cdot, \infty)$  be the Green function in  $D_0$ . Then the Jordan arcs  $J_j$ ,  $j \in I$ , are trajectories of the quadratic differential  $q(z)dz^2$  with  $q$  defined by*

$$q(z) := \left( 2 \frac{\partial}{\partial z} g_{D_0}(z, \infty) \right)^2, \quad (5.3)$$

where  $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$  is the usual complex differentiation. The function  $q$  has a meromorphic (single-valued) continuation throughout the domain  $\overline{\mathbb{C}} \setminus E'_0$  with  $E'_0$  denoting the sets of cluster points of  $E_0$ . Near infinity we have

$$q(z) = \frac{1}{z^2} + O(z^{-3}) \quad \text{as } z \rightarrow \infty. \quad (5.4)$$

The function  $q$  has at most simple poles in isolated points of  $E_0$ , and it is analytic throughout  $\overline{\mathbb{C}} \setminus E_0$ .

It is not difficult to verify that the meromorphy of the function  $q$  in  $\overline{\mathbb{C}} \setminus E'_0$  is equivalent to the  $S$ -property (5.1).

The local structure of the trajectories of quadratic differentials can rather easily be understood and described (for more details see Subsection 11.5, further below). Of special interest are neighborhoods of poles and zeros of the function  $q$  in (5.3).

**REMARK 2.** *Since we know from Theorem 8 that all Jordan arcs  $J_j$ ,  $j \in I$ , are trajectories of a quadratic differential  $q(z)dz^2$  that is meromorphic in  $\overline{\mathbb{C}} \setminus E'_0$ , it follows from the local structure of the trajectories that all Jordan arcs  $J_j$ ,  $j \in I$ , that end at an isolated point  $z$  of  $E_0 \cup E_1$  form a regular star at this point.*

**5.3. Rational Quadratic Differentials.** The description of the Jordan arcs  $J_j$ ,  $j \in I$ , as trajectories of a quadratic differential  $q(z)dz^2$  is especially constructive if the function  $q$  in (5.3) is rational. This is the case if the set  $E_0$  from Theorem 4 is finite. Algebraic functions  $f$  are prototypical examples for this situation.

For the formulation of the main result in this direction, we need the notion of bifurcation points in  $K_0(f, \infty)$ , the associated bifurcation index, and the notion of critical points of the Green function  $g_{D_0}(\cdot, \infty)$ .

**DEFINITION 6.** *Let the objects  $K_0 = K_0(f, \infty)$ ,  $E_0, E_1 \subset \overline{\mathbb{C}}$ , and  $\{J_j\}_{j \in I}$  be and ones as in the Theorems 4 or 8. For each isolated point  $z \in E_1 \cup E_0$ , the bifurcating index  $i(z)$  is the number of different Jordan arcs  $J_j$ ,  $j \in I$ , that end at this point  $z$ .*

If  $z$  is an isolated point of  $K_0 = K_0(f, \infty)$ , then  $z$  lies necessarily in  $E_0$ , and by definition we have  $i(z) = 0$  since  $z$  has no contact to any arc in  $K_0$ . Such isolated points can exist; they are generated by isolated, essential singularities of the function  $f$  that are no branch points.

**DEFINITION 7.** *Let  $D_0 = D_0(f, \infty)$  be the extremal domain, and assume that  $\text{cap}(K_0(f, \infty)) > 0$ . By  $E_2 \subset D_0$  we denote the set of all critical points of the Green function  $g_{D_0}(z, \infty)$ , and for each  $z \in E_2$  we denote the order of the critical point  $z$  by  $j(z)$ , i.e., for  $z \in E_2$ , we have*

$$\frac{\partial^l}{\partial z^l} g_{D_0}(z, \infty) \begin{cases} = 0 & \text{for } l = 1, \dots, j(z) \\ \neq 0 & \text{for } l = j(z) + 1. \end{cases} \quad (5.5)$$

If  $\text{cap}(K_0(f, \infty)) = 0$ , then we set  $E_2 = \emptyset$ .

The sets  $E_1$  and  $E_2$  are always discrete in  $\overline{\mathbb{C}} \setminus E_0$ , while the set  $E_0$  can be a mixture of isolated and cluster points. Because of this later possibility, it was necessary to distinguish the set  $E'_0$  of cluster points from the original set  $E_0$  in Theorem 8. The set  $E_1 \cup E_0 \setminus E'_0$  contains all isolated points of  $E_1 \cup E_0$ . We have  $E'_0 = \emptyset$  if and only if  $E_0$  is finite.

**PROPOSITION 3.** *If  $E_0$  is a finite set, then the sets  $E_1$  and  $E_2$  are necessarily also finite.*

After these preliminaries, we are ready to formulate the central results of the present subsection.

**THEOREM 9.** *We use the same notations as in the Theorems 4 and 8, and assume that the set  $E_0$  is finite. Then the function  $q$  in (5.3) is rational, and we have the explicit representation*

$$q(z) = \prod_{v \in E_0 \cup E_1, i(v) > 0} (z - v)^{i(v)-2} \prod_{v \in E_2} (z - v)^{2j(v)}. \quad (5.6)$$

Notice that there always exist points  $z \in E_0$  with  $i(z) = 1$ , which implies that  $q$  always is a broken rational function. Actually, this assertion follows already from (5.4) in Theorem 8, and further we deduce from (5.4) that the denominator degree of  $q$  is exactly 2 degrees larger than its numerator degree.

The explicit formula (5.6) for  $q$  can be very helpful for the numerical calculation of the analytic Jordan arcs  $J_j$ ,  $j \in I$ , in  $K_0(f, \infty)$ . If the points of the sets  $E_0$ ,  $E_1$ , and  $E_2$  have been determined, then most of the work is done, and one can calculate the Jordan arcs  $J_j$ ,  $j \in I$ , by solving a differential equation that is based on (5.2), (5.3), and (5.6). This procedure has, for instance, also been used for the calculation of the arcs in the minimal sets  $K_0(f_j, \infty)$ ,  $j = 1, \dots, 5$ , in the Examples 6.1 - 6.5 that follow next. The critical part of the job is the calculation of the zeros of the function  $q$  in (5.6). More information about this topic can be found at the end of the discussion of Example  $f_3$  in Subsection 6.3.

## 6. Examples

In the present section we consider five specially chosen algebraic functions  $f = f_1, \dots, f_5$ , and discuss for each of them the solution of Problem  $(f, \infty)$ . Typically, we calculate and plot the minimal set  $K_0(f, \infty)$ , discuss particular features of its shape, and identify the sets  $E_0$ ,  $E_1$ ,  $E_2$ , and the family of Jordan arcs  $J_j$ ,  $j \in I$ , that have been introduced in Theorem 4 and in Definition 7. Also the quadratic differential  $q(z)dz^2$  from Theorem 9 is identified for each case.

Some of the examples depend on one or two parameters; and variations of these parameters will be done in order to understand the mechanisms that lead to special features of the minimal set  $K_0(f, \infty)$ . Of special interest are:

- a) The connectivity of the minimal set  $K_0(f, \infty)$  together with the question of how it changes under variations of the function  $f$ .
- b) The identification of active versus inactive branch points of  $f$ . It turns out that in general not all branch points of the function  $f$  play an active role in the determination of the minimal set  $K_0(f, \infty)$ , and for the calculation of  $K_0(f, \infty)$  it is important to know already in advance which of them are active and which ones remain passive.

The presentation and discussion of the five examples demands comparatively much space, and there has been some hesitation to include all the material. But it is hoped that the expenses on space and efforts are counterbalanced by an improved understanding of the definitions and results presented in the last four sections.

**6.1. Example  $f_1$ .** As a first, and in most aspects rather trivial example, we consider the function

$$f_1(z) := \frac{1}{\sqrt{z^2 - 1}}, \quad (6.1)$$

which often appears in approximation theory, and has been included here as a warm-up exercise.

Clearly, the function has branch points at  $-1$  and  $1$ . Therefore, the set  $\mathcal{D}(f_1, \infty)$  of admissible domains for Problem  $(f_1, \infty)$  from Definition 1 consists of all domains  $D \subset \overline{\mathbb{C}}$  such that  $\infty \in D$  and that the two points  $-1$  and  $1$  are connected in the complement  $K = \overline{\mathbb{C}} \setminus D$ . The uniquely existing extremal domain of Theorem 2 is given by

$$D_0(f_1, \infty) = \overline{\mathbb{C}} \setminus [-1, 1], \quad (6.2)$$

and the minimal set by  $K_0(f_1, \infty) = [-1, 1]$ . As sets  $E_0$ ,  $E_1$ ,  $E_2$ , and arcs  $J_j$ ,  $j \in I$ , introduced in Theorem 4 and in Definition 7, we have  $E_0 = \{-1, 1\}$ ,  $E_1 = E_2 = \emptyset$ ,  $I = \{1\}$ , and  $J_1 = (-1, 1)$ . Solution (6.2) is a consequence of the monotonicity of  $\text{cap}(\cdot)$  under projections onto straight lines (cf., Lemma 22 in Subsection 11.1,

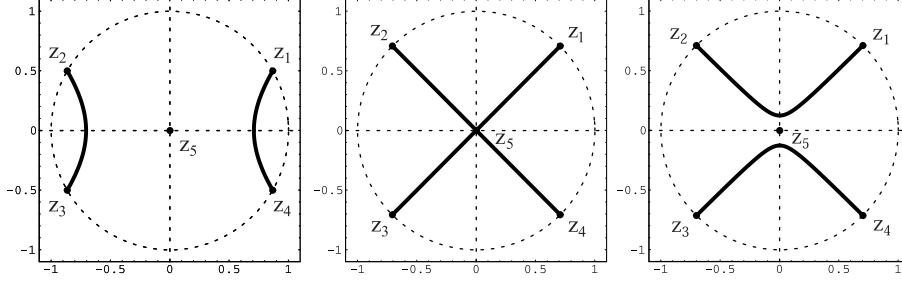


FIGURE 2. Three examples of minimal sets  $K_0(f_2, \infty)$  for Problem  $(f_2, \infty)$  with  $f_2$  defined in (6.5). The three windows corresponded to the three parameter values  $\varphi = \pi/6$ ,  $\varphi = \pi/4$ , and  $\varphi = 101\pi/400$ , respectively.

further below). The single arc  $J_1 = (-1, 1)$  in  $K_0(f_1, \infty)$  is a trajectory of the quadratic differential

$$\frac{1}{z^2 - 1} dz^2, \quad (6.3)$$

i.e., it satisfies the relation

$$\frac{1}{z^2 - 1} dz^2 < 0, \quad (6.4)$$

and (6.3) corresponds to Theorem 9.

**6.2. Example  $f_2$ .** Next, we consider a function  $f_2$  that depends on a parameter  $\varphi$ . For  $\varphi \in (0, \pi/2)$ , we define  $(\varphi_j)_{j=1,\dots,4} := (\varphi, \pi - \varphi, \pi + \varphi, 2\pi - \varphi)$ ,  $z_j := \exp(i\varphi_j)$ ,  $j = 1, \dots, 4$ ,  $P_4(z) := \prod_{j=1}^4 (1 - z_j/z)$ , and then we define the function  $f_2$  as

$$f_2(z) := \sqrt[3]{P_4(z)} \quad (6.5)$$

with a choice of the sign of the square root in (6.5) so that  $f_2(\infty) = 1$ . The function  $f_2$  has the four branch points  $z_1, \dots, z_4$ , and it is symmetric with respect to the real and the imaginary axis. The symmetries lead to corresponding symmetries of the minimal set  $K_0(f_2, \infty)$  and the extremal domain  $D_0(f_2, \infty)$  for each  $\varphi \in (0, \pi/2)$ .

For each  $\varphi \in (0, \pi/2)$ , the set  $\mathcal{D}(f_2, \infty)$  of admissible domains introduced in Definition 1 consists of all domains  $D \subset \mathbb{C}$  such that  $\infty \in D$  and that at least two disjoint pairs of the four branch points  $z_1, \dots, z_4$  are connected in  $K = \mathbb{C} \setminus D$ . It is not necessary that all four points  $z_1, \dots, z_4$  are connected, nor that a specific combination of pairs has to be connected in  $K = \mathbb{C} \setminus D$ .

From the uniqueness of the minimal set  $K_0(f_2, \infty)$ , which has been proved in Theorem 2, it follows that from the variety of connectivities that are possible for the set  $K \in \mathcal{K}(f_2, \infty)$  and a given fixed parameter value  $\varphi$ , a specific one is selected as the minimal set  $K_0(f_2, \infty)$ .

The shape and the connectivity of the minimal set  $K_0(f_2, \infty)$  depends on the parameter  $\varphi$ , and we distinguish the three cases  $0 < \varphi < \pi/4$ ,  $\varphi = \pi/4$ , and  $\pi/4 < \varphi < \pi/2$ , which we will label as cases  $a$ ,  $b$ , and  $c$ , respectively. In the three windows of Figure 2, the three cases are represented by the minimal sets  $K_0(f_2, \infty)$  for the parameter values  $\varphi = \pi/6$ ,  $\varphi = \pi/4$ , and  $\varphi = 101\pi/400$ , respectively. The

value  $\varphi = 101\pi/400$  has been chosen to be close to the critical value  $\varphi = \pi/4$ . The picture in the third window gives an impression of the metamorphosis of the set  $K_0(f_2, \infty)$  when  $\varphi$  approaches and then crosses the critical value  $\varphi_0 = \pi/4$ .

In the two cases  $a$  and  $c$ , the minimal set  $K_0(f_2, \infty)$  consists of two components. We have  $E_0 = \{z_1, \dots, z_4\}$ ,  $E_1 = \emptyset$ ,  $E_2 = \{0\}$ ,  $I = \{1, 2\}$ , and the two analytic Jordan arcs  $J_1$  and  $J_2$  in  $K_0(f_2, \infty)$  which connect the two pairs of branch points  $\{z_1, z_4\}$  and  $\{z_2, z_3\}$  in case  $a$  and the two pairs  $\{z_1, z_2\}$  and  $\{z_3, z_4\}$  in case  $c$ .

The case  $b$  corresponds to the single parameter value  $\varphi = \pi/4$ . Here, all four branch points  $z_1, \dots, z_4$  are connected in  $K_0(f_2, \infty)$ ; the set is a continuum. We have  $E_0 = \{z_1, \dots, z_4\}$ ,  $E_1 = \{0\}$ ,  $E_2 = \emptyset$ ,  $I = \{1, \dots, 4\}$ , and the four Jordan arcs  $J_1, \dots, J_4$  in  $K_0(f_2, \infty)$  are the four segments  $(0, z_j)$ ,  $j = 1, \dots, 4$ .

The two Jordan arcs  $J_1$  and  $J_2$  in the two cases  $a$  and  $c$ , and also the 4 Jordan arcs  $J_1, \dots, J_4$  in case  $b$ , are trajectories of the quadratic differential

$$\frac{z^2}{\prod_{j=1}^4 (z - z_j)} dz^2. \quad (6.6)$$

Taking advantage of the symmetry of the function  $f_2$ , one can show that for each  $\varphi \in (0, \pi/2) \setminus \{\pi/4\}$  the two arcs  $J_1$  and  $J_2$  are sections of an hyperbole. Indeed, it is not difficult to verify that the mapping  $z \mapsto z^2$  maps the two arcs  $J_1$  and  $J_2$  onto one straight segment, which proves this last assertion.

**6.3. Example  $f_3$ .** The third example is very similar to the second one, only that now the forth root is taken instead of the square root in (6.5). We use the same definitions for  $\varphi$ ,  $\varphi_j$ ,  $z_j$ ,  $j = 1, \dots, 4$ , and  $P_4$  as in Example 6.2, and define function  $f_3$  as

$$f_3(z) := \sqrt[4]{P_4(z)}. \quad (6.7)$$

The branch of the root  $\sqrt[4]{\cdot}$  is chosen so that  $f_3(\infty) = 1$ . Although the basic structure of the two functions  $f_3$  and  $f_2$  is very similar, there exist decisive differences with respect to their meromorphic continuability. For each parameter value  $\varphi \in (0, \pi/2)$ , the set  $\mathcal{D}(f_3, \infty)$  of admissible domains for Problem  $(f_3, \infty)$  consists of all domains  $D \subset \mathbb{C}$  such that  $\infty \in D$  and all four branch points  $z_1, \dots, z_4$  are connected in the complementary set  $K = \overline{\mathbb{C}} \setminus D$ .

As in Example 6.2, we distinguish three cases  $a$ ,  $b$ , and  $c$ , which are again defined by  $0 < \varphi < \pi/4$ ,  $\varphi = \pi/4$ , and  $\pi/4 < \varphi < \pi/2$ , respectively. In all three cases, the minimal set  $K_0(f_3, \infty)$  is connected, and the extremal domain  $D_0(f_3, \infty)$  is simply connected. However, the minimal set  $K_0(f_3, \infty)$  is of a somewhat different structure in each of the three cases.

In case  $b$ , the two functions  $f_2$  and  $f_3$  have an identical extremal domain  $D_0(f_3, \infty)$  and an identical minimal set  $K_0(f_3, \infty) = K_0(f_2, \infty)$ . The minimal set has already been shown in the middle window of Figure 2.

For the two other cases  $a$  and  $c$ , two representatives of the minimal sets  $K_0(f_3, \infty)$  are shown in Figure 3. The two cases are represented by the same two parameter values  $\varphi = \pi/6$  and  $\varphi = 101\pi/400$  as already used before in Figure 2. A new phenomenon now is the appearance of two bifurcation points in  $K_0(f_3, \infty)$ , which are denoted by  $z_5$  and  $z_6$  in Figure 3.

In the two cases  $a$  and  $c$ , we have  $E_0 = \{z_1, \dots, z_4\}$ ,  $E_1 = \{z_5, z_6\}$ ,  $E_2 = \emptyset$ ,  $I = \{1, \dots, 5\}$ , and the five open analytic Jordan arcs  $J_1, \dots, J_5$  in  $K_0(f_3, \infty)$  connect the six points  $z_1, \dots, z_6$  as shown in Figure 3. These five Jordan arcs  $J_1, \dots$ ,

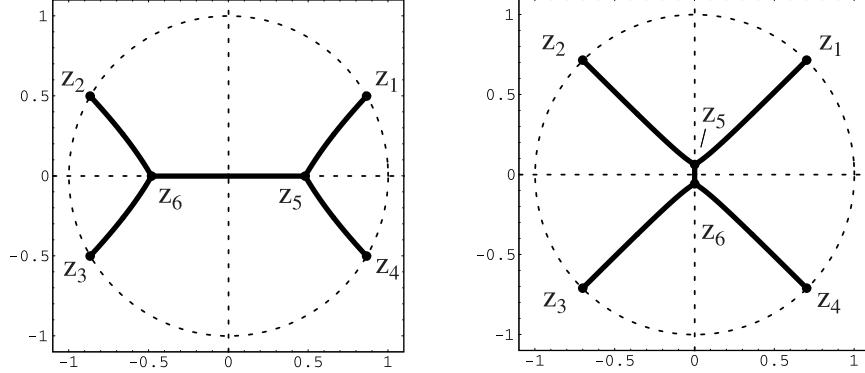


FIGURE 3. Two examples of minimal sets  $K_0(f_3, \infty)$  for Problem  $(f_3, \infty)$  with  $f_3$  defined in (6.7). The two windows correspond to the parameter values  $\varphi = \pi/6$  and  $\varphi = 101\pi/400$ , respectively.

$J_5$ , and also the four arcs in case  $b$ , are trajectories of the quadratic differential

$$\frac{(z - z_5)(z - z_6)}{\prod_{j=1}^4 (z - z_j)} dz^2. \quad (6.8)$$

Notice that in case  $b$ , we have  $z_5 = z_6 = 0$ . In the two other cases, we always have  $z_5 = -z_6 \neq 0$ . From a practical point of view the calculation of the two bifurcation points  $z_5$  and  $z_6$  is the main work and causes the main difficulties for the calculation of the arcs  $J_1, \dots, J_5$ . We want to take a closer look on this problem.

The form of the quadratic differential (6.8) already suggests that elliptic integrals should play a role in the analytic determination of the bifurcation points  $z_5$  and  $z_6$ . Indeed, with the machinery presented in [18], [3], or [22], it is not too difficult to formulate conditions that allow to determine the points  $z_5$  and  $z_6$ . We reproduce the main elements of the procedure for case  $a$ , i.e., for the case  $\varphi \in (0, \pi/4)$ , and define the function

$$g(a, x) := \left| \sqrt{\frac{x - a}{x(x^2 - 2x \cos(2\varphi) + 1)}} \right| \quad \text{for } a \in (0, 1), \quad x \in \mathbb{R}. \quad (6.9)$$

The improper elliptical integral

$$I(a) := \lim_{c \rightarrow +\infty} \left[ \int_a^c g(a, x) dx - \int_{-c}^0 g(a, x) dx \right] \quad (6.10)$$

is strictly monotonic for  $a \in (0, 1)$ , and we have  $I(0) > 0$  and  $I(1) < 0$ . Consequently, there uniquely exists  $a_0 \in (0, 1)$  with  $I(a_0) = 0$ . The two bifurcation points  $z_5$  and  $z_6$  are then given by

$$z_5 = z_5(\varphi) = +\sqrt{a_0}, \quad z_6 = -z_5. \quad (6.11)$$

For the special parameter value  $\varphi = \pi/6$ , for which the corresponding minimal set  $K_0(f_3, \infty)$  is shown in the first window of Figure 3, we get

$$a_0 = 0.231584, \quad z_5 = 0.481232, \quad \text{and} \quad z_6 = -0.481232. \quad (6.12)$$



In a derivation of the expressions (6.9) and (6.10), one has in a first step to transform the minimal set  $K_0(f_3, \infty)$  by the mapping  $z \mapsto z^2$  into a continuum that connects the three points  $0$ ,  $e^{i2\varphi}$ , and  $e^{-i2\varphi}$ .

After the reduction to a three-point problem, one can apply results that have been proved in [12] (see also [13], Theorem 1.5). In [13], Theorem 1.5, the value  $a_0$  is expressed as the solution of a system of four equations that involve Jacobi elliptical functions and theta functions. We have not investigated whether the approach is numerically easier to handle than the equation  $I(a_0) \stackrel{!}{=} 0$ , which is based on (6.10). In any case, the level of difficulties that arise already in this rather simply structured case of function  $f_3$  gives an idea of the type of difficulties that arise if one has to determine the points in the set  $E_1$  (and  $E_2$ ) in a more general situation. In the next two examples these points have been calculated by a numerical method that has been developed by the author on an ad-hoc basis. It is based on a geometrical approach. The method will be published in a separate paper. Further comments about the numerical side of the problem will be made in Subsection 8.3, further below.

**6.4. Example  $f_4$ .** In the fourth example, we consider a modification of the function  $f_3$ , which itself has already been a modification of function  $f_2$ . We use again the definitions  $\varphi$ ,  $\varphi_j$ ,  $z_j$ ,  $j = 1, \dots, 4$ , and  $P_4$  from Example 6.2, and define the new function  $f_4$  as

$$f_4(z) := \sqrt[2]{\sqrt[2]{P_4(z)} - c}. \quad (6.13)$$

In addition to the former parameter  $\varphi$ , there is now a second parameter  $c$ , which may assume arbitrary complex values  $c \in \mathbb{C}$ , but we shall consider only special situations. We discuss complex values of  $c$  that lie near the origin, and in addition real values of  $c$  in the interval  $(0, 1)$ . The signs of the inner and outer square root in (6.13) are assumed to be chosen in such a way that both roots are positive for  $z = \infty$  and  $c = 0$ . In case of  $c = 0$ , the two functions  $f_4$  and  $f_3$  are identical.

The study of the function  $f_4$  and its associated minimal set  $K_0(f_4, \infty)$  will be more complex and involved than that of the last two examples, which in some sense have been preparations of the present example. Our main interest will be concentrated on the following three questions:

- 1) It is not difficult to see that for almost all parameter constellations the function  $f_4$  has 8 branch points. But not all of them will always play an active role in the determination of the minimal set  $K_0(f_4, \infty)$ , some of them are hidden away from  $K_0(f_4, \infty)$  somewhere on a 'lower' sheet of the Riemann surface  $\mathcal{R}_{f_4}$  that is defined by  $f_4$ . In the terminology of Section 3, we can say that these inactive branch points on  $\mathcal{R}_{f_4}$  stay away from the extremal domain  $D_0(\mathcal{R}_{f_4}, \infty^{(0)}) \subset \mathcal{R}_{f_4}$ . The first question in our discussion is therefore: Which of the branch points of  $f_4$  are 'active' and which ones are 'inactive' for a given parameter constellation?
- 2) We have already seen in Example 6.2 that the connectivity of the minimal set  $K_0(f_4, \infty)$  can change. Motivated by this experience, the second question will be: What is the connectivity of the minimal set  $K_0(f_4, \infty)$  for a given parameter constellation, and how does it change with variations of the parameter values?
- 3) At the end of the last example we have discussed in some detail the difficulties to find the points of the set  $E_1$ . In general these points are bifurcation

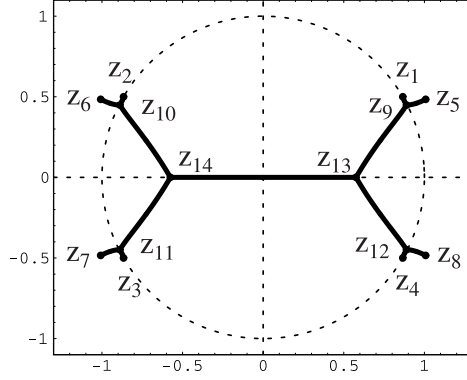


FIGURE 4. The minimal set  $K_0(f_4, \infty)$  for Problem  $(f_4, \infty)$  with  $f_4$  defined in (6.13) with parameter values  $\varphi = \pi/6$  and  $c = \sqrt{0.4}$ .

points of the minimal set, and these points are crucial for the quadratic differential (5.6) in Theorem 9. The third question is therefore: How do the bifurcation points of the minimal set  $K_0(f_4, \infty)$  depend on the parameter values, and at which parameter constellations do these points merge or split up?

The function  $f_4$  has in general eight branch points; four of them are identical with those of the two functions  $f_2$  and  $f_3$ , and they will be denoted again by  $z_1, \dots, z_4$ . These four branch points do not depend on the parameter  $c$ .

For every parameter  $\varphi \in [0, \pi/2)$  there exists a whole region of parameter values  $c$  such that only these four 'old' branch points  $z_1, \dots, z_4$  of  $f_4$  appear in the minimal set  $K_0(f_4, \infty)$ , and in these cases they are the only branch points that play an active role in the determination of  $K_0(f_4, \infty)$ . All other branch points will be called 'inactive'.

Throughout the discussion, we keep the parameter  $\varphi = \pi/6$  fixed, which implies that all minimal sets  $K_0(f_4, \infty)$  that will be considered during our discussion should be compared with the set  $K_0(f_3, \infty)$  in the first window of Figures 3.

In a first step we choose

$$c = r e^{it} \quad \text{with } t \in [0, 2\pi) \quad \text{and } r > 0 \text{ small,} \quad (6.14)$$

and see what happens. If  $|c| > 0$  is small, then the four new branch points  $z_5, \dots, z_8$  of the function  $f_4$  lie close to the four old branch points  $z_1, \dots, z_4$ . In Figure 4 the situation is shown for the parameter values  $\varphi = \pi/6$  and  $c = \sqrt{0.4}$ . Of course,  $\sqrt{0.4}$  is not very small, however, smaller values of  $|c|$  lead to configurations that are difficult to plot.

While in (6.14) the parameter  $t$  runs through  $[0, 2\pi)$ , each one of the four new branch points  $z_5, \dots, z_8$  encircles two times the corresponding old branch point  $z_1, \dots, z_4$ .

The interesting point is now that the four new branch points  $z_5, \dots, z_8$  are elements of the minimal set  $K_0(f_4, \infty)$  only on one half of their twofold circular path. On the other half, they become 'inactive', i.e., they are hidden away on another sheet of the Riemann surface  $\mathcal{R}_{f_4}$ . In this later case, the set  $K_0(f_4, \infty)$

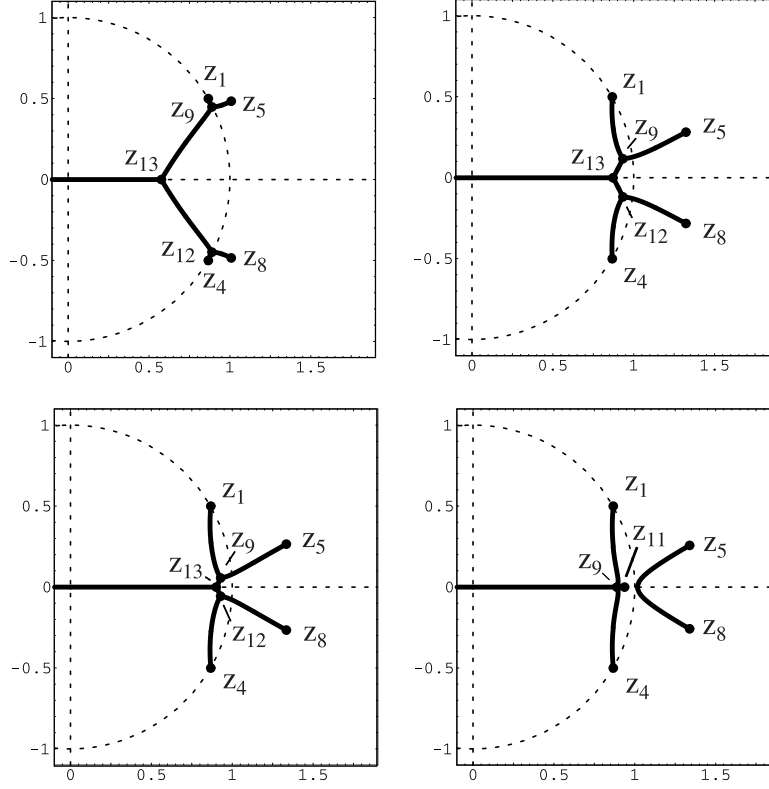


FIGURE 5. Four examples of minimal sets  $K_0(f_4, \infty)$  for Problem  $(f_4, \infty)$  with  $f_4$  defined in (6.13) with the parameter  $\varphi = \pi/6$  fixed and parameter values  $c = \sqrt{0.4}, \sqrt{0.7}, \sqrt{0.705}$ , and  $\sqrt{0.715}$  from top row left to bottom row right.

contains only the four branch points  $z_1, \dots, z_4$ , and consequently, it is identical with the minimal set  $K_0(f_3, \infty)$ , which has been shown in the first window of Figure 3.

It has already been said that in Figure 4, the minimal set  $K_0(f_4, \infty)$  is shown for the parameter values  $\varphi = \pi/6$  and  $c = \sqrt{0.4}$ . This is a parameter constellation in which all eight branch points  $z_1, \dots, z_8$  are active. In contrast to this, the parameter constellation  $\varphi = \pi/6$  and  $c = -\sqrt{0.4}$ , which corresponds to  $t = \pi$  in (6.14), leads to a minimal set  $K_0(f_4, \infty)$  that contains only the four old branch points  $z_1, \dots, z_4$ , and it is therefore identical with the minimal set  $K_0(f_3, \infty)$  shown in the first window of Figure 3.

Studying the minimal set  $K_0(f_4, \infty)$  for  $|c|$  small, gives a good illustration of the phenomenon of active and inactive branch points. Of course, an extension of such a discussion to arbitrary values of  $c \in \mathbb{C}$  would be possible, but it becomes rather complicated.

Next, we consider Problem  $(f_4, \infty)$  for the six specially chosen real parameter values  $c = \sqrt{0.4}, \sqrt{0.7}, \sqrt{0.705}, \sqrt{0.715}, \sqrt{0.74}, \sqrt{0.76}$  and keep again  $\varphi = \pi/6$  fixed. The selected values should be seen as representatives for the general situation of  $c \in (0, 1)$ . The discussion will show why the specific selection is interesting.

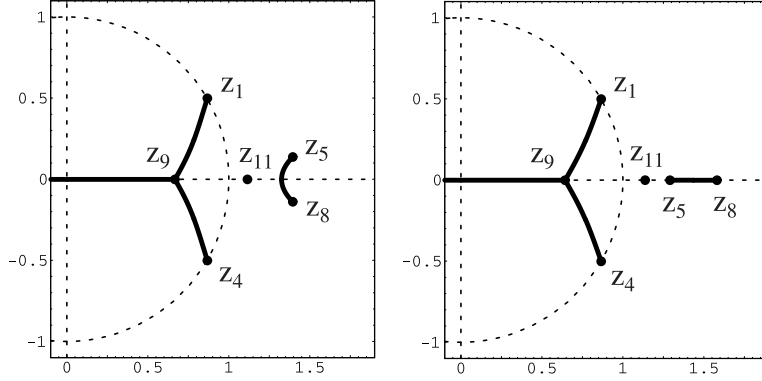


FIGURE 6. Two examples of minimal sets  $K_0(f_4, \infty)$  for Problem  $(f_4, \infty)$  with  $f_4$  defined in (6.13) with parameter  $\varphi = \pi/6$  and  $c = \sqrt{0.74}$  in the left window and  $\varphi = \pi/6$  and  $c = \sqrt{0.76}$  in the right window.

There exists numerical evidence (but no analytic proof, so far) that at the critical parameter value  $c_0 = \sqrt[4]{1/2}$ , the minimal set  $K_0(f_4, \infty)$  changes its connectivity. It is obvious that there exists  $c_0 \in (0, 1)$  which is equal to, or lies close to  $\sqrt[4]{1/2}$  such that for  $0 \leq c \leq c_0$  the set  $K_0(f_4, \infty)$  is connected, and for  $c_0 < c < 1$  it is disconnected. In the disconnected case, it consists of three components. For  $c \rightarrow c_0 - 0$ , in each of the two half-planes  $\{\operatorname{Re}(z) \leq 0\}$  three bifurcation points of  $K_0(f_4, \infty)$  merge and form a new bifurcation point of order five in each of the two half-planes.

In Figure 5, the sequence of four minimal sets  $K_0(f_4, \infty)$  is shown for the parameter values we have  $c = \sqrt{0.4}, \sqrt{0.7}, \sqrt{0.705}, \sqrt{0.715}$ . The sequence shows the metamorphosis of the set  $K_0(f_4, \infty)$  while the parameter  $c$  crosses the critical value  $c_0 = \sqrt[4]{1/2} = \sqrt{0.707\dots}$ . In the four windows the set  $K_0(f_4, \infty)$  is shown only for the right half-plane.

In the first three windows of Figure 5, the minimal set  $K_0(f_4, \infty)$  is connected, and there are three bifurcation points  $z_9, z_{12}$ , and  $z_{13}$ , each of order 3, which then merge to a single bifurcation point when  $c$  reaches the critical value  $c_0$ . At that moment, the new bifurcation point is of order 5.

When the critical value  $c_0$  has been passed, then the minimal set  $K_0(f_4, \infty)$  is disconnected, as shown in the fourth window of Figure 5. There remains a bifurcation point  $z_9$  of order 3, and as a new phenomenon, we have a critical point of the Green function  $g_{D_0}(\cdot, \infty)$ ,  $D_0 = D_0(f_4, \infty)$ , at  $z_{11}$ .

Another interesting parameter value is  $c_1 = \sqrt{3/4}$ , since at the parameter constellation  $\varphi = \pi/6$  and  $c_1 = \sqrt{3/4}$  two pairs of branch points of the function  $f_4$  collapse to simple zeros of  $f_4$ . These two simple zeros are located at  $\pm\sqrt{2}$ .

In Figure 6, the transition process at the critical value  $c_1 = \sqrt{3/4}$  is represented by the two parameter values  $c = \sqrt{0.74}$  and  $c = \sqrt{0.76}$ . One can see how the concerned components of  $K_0(f_4, \infty)$  change their shape from a type of vertical arcs to horizontal slits.

We conclude the discussion of Example 6.4 by assembling informations about the sets  $E_0$ ,  $E_1$ ,  $E_2$ , and the arcs  $J_j$ ,  $j \in I$ , introduced in Theorem 4 and in Definition 7. This is done for the six parameter constellations of the two Figures 5 and 6. In addition we also give the quadratic differential  $q(z)dz^2$  from Theorem 9. This information corresponds to the whole set  $K_0(f_4, \infty)$ , while in the Figures 5 and 6 only restrictions to the right half-plane have been plotted.

For the three parameter values  $c = \sqrt{0.4}$ ,  $\sqrt{0.7}$ ,  $\sqrt{0.705}$  the minimal set  $K_0(f_4, \infty)$  is connected, and with respect to  $E_0$ ,  $E_1$ ,  $E_2$ ,  $J_j$ ,  $j \in I$ , and the quadratic differential  $q(z)dz^2$  we have identical structures.

We have  $E_0 = \{z_1, \dots, z_8\}$ ,  $E_1 = \{z_9, \dots, z_{14}\}$ , and  $E_2 = \emptyset$ . All eight branch points  $z_1, \dots, z_8$  of  $f_4$  are active, there are six bifurcation points  $z_9, \dots, z_{14}$  and 13 Jordan arcs  $J_j$ ,  $j \in I = \{1, \dots, 13\}$ . In accordance to Theorem 9, all 13 arcs  $J_j$ ,  $j \in I$ , are trajectories of the quadratic differential

$$\frac{\prod_{j=9}^{14}(z - z_j)}{\prod_{j=1}^8(z - z_j)}dz^2. \quad (6.15)$$

For the three parameter values  $c = \sqrt{0.715}$ ,  $\sqrt{0.74}$ ,  $\sqrt{0.76}$ , which correspond to the fourth window in Figure 5 and the two windows in Figure 6, the minimal set  $K_0(f_4, \infty)$  consists of three components. The sets  $E_0$ ,  $E_1$ ,  $E_2$ ,  $J_j$ ,  $j \in I$ , and the quadratic differential  $q(z)dz^2$  are of the same structure in all three cases. There are two bifurcation points  $z_9, z_{10}$ , and the Green function  $g_{D_0}(\cdot, \infty)$ ,  $D_0 = D_0(f_4, \infty)$ , has two critical points  $z_{11}$  and  $z_{12}$ .

Thus, we have  $E_0 = \{z_1, \dots, z_8\}$ ,  $E_1 = \{z_9, z_{10}\}$ , and  $E_2 = \{z_{11}, z_{12}\}$ . There are 7 Jordan arcs  $J_j$ ,  $j \in I = \{1, \dots, 7\}$ , and these arcs are trajectories of the quadratic differential

$$\frac{\prod_{j=9}^{10}(z - z_j) \prod_{j=11}^{12}(z - z_j)^2}{\prod_{j=1}^8(z - z_j)}dz^2. \quad (6.16)$$

**6.5. Example  $f_5$ .** As a last example, we come back to the algebraic function (1.1), which has already been used in the Introduction for a demonstration of the connection between Padé approximation and sets of minimal capacity. This function is now denoted as  $f_5$ , and it has been defined in (1.1) as

$$f_5(z) := \sqrt[4]{\prod_{j=1}^4(1 - z_j/z)} + \sqrt[3]{\prod_{j=5}^7(1 - z_j/z)} \quad (6.17)$$

with the 7 branch points that have been chosen as

$$\begin{aligned} z_1 &= 1 + 3i, & z_2 &= -4 + 2i, & z_3 &= -4 + i, & z_4 &= 0 + 2i, \\ z_5 &= 2 + 2i, & z_6 &= 3 + 4i, & z_7 &= 1 + 4i. \end{aligned} \quad (6.18)$$

The choice of the branch points was in principle arbitrary, but it reflects the intension to avoid symmetries in the minimal set  $K_0(f_5, \infty)$  of a sort that has been very dominant in the 3 Examples 6.2 - 6.4.

From the structure of function  $f_5$ , we conclude that the set  $\mathcal{D}(f_5, \infty)$  of admissible domains for Problem  $(f_5, \infty)$  introduced in Definition 1 consists of all domains  $D \subset \overline{\mathbb{C}}$  such that  $\infty \in D$  and that the elements of each of the two subsets of branch points  $\{z_1, \dots, z_4\}$  and  $\{z_5, z_6, z_7\}$  are connected in the complement  $K = \overline{\mathbb{C}} \setminus D$ .

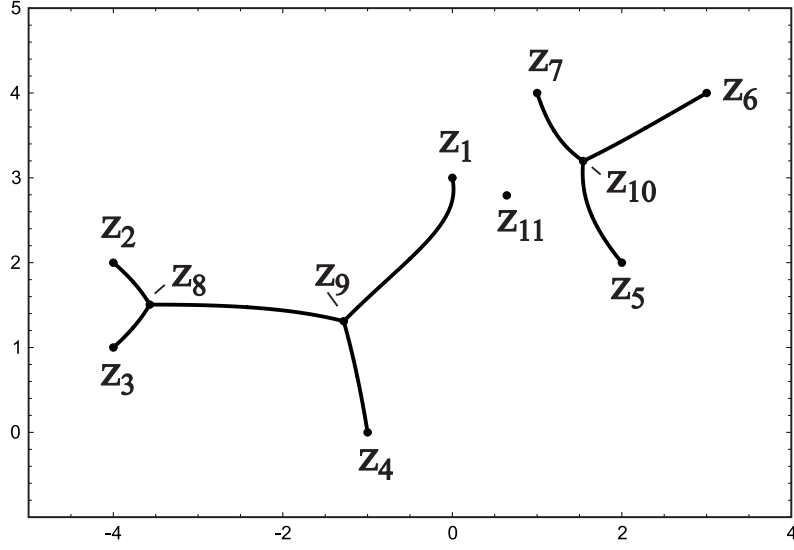


FIGURE 7. The minimal sets  $K_0(f_5, \infty)$  for Problem  $(f_5, \infty)$  with  $f_5$  defined by (6.17) and (6.18).

It turns out that the minimal set  $K_0(f_5, \infty)$  consists of two components, and that indeed each of them connects one of the two sets  $\{z_1, \dots, z_4\}$  and  $\{z_5, z_6, z_7\}$ . The set  $K_0(f_5, \infty)$  is shown in Figure 7. It has three bifurcation points, which are denoted by  $z_8, z_9, z_{10}$ , and the Green function  $g_{D_0}(\cdot, \infty)$  in the extremal domain  $D_0 = D_0(f_5, \infty)$  possesses exactly one critical point, which is denoted by  $z_{11}$  in Figure 7.

While the 7 branch points  $z_1, \dots, z_7$  in (6.18) can be considered as input to the problem, the location of the four other points  $z_8, \dots, z_{11}$  has to be determined by the criterion of minimality of the set  $K_0(f_5, \infty)$ . The calculation of these four points has been done numerically, and their values are

$$\begin{aligned} z_8 &= -3.57021 + 1.50570i, \\ z_9 &= -1.28112 + 1.30991i, \\ z_{10} &= 1.54341 + 3.19816i, \\ z_{11} &= 0.64231 + 2.79311i. \end{aligned} \tag{6.19}$$

The 8 Jordan arcs  $J_j$ ,  $j \in I = \{1, \dots, 8\}$ , in  $K_0(f_5, \infty)$  are trajectories of the quadratic differential

$$\frac{(z - z_{11})^2 \prod_{j=8}^{10} (z - z_j)}{\prod_{j=1}^7 (z - z_j)} dz^2. \tag{6.20}$$

The sets  $E_0, E_1, E_2$  introduced in Theorem 4 and in Definition 7 are now  $E_0 = \{z_1, \dots, z_7\}$ ,  $E_1 = \{z_8, z_9, z_{10}\}$ , and  $E_2 = \{z_{11}\}$ .

**6.6. Some General Remarks.** The main motivation for the selection and presentation of the 5 Examples 6.1 - 6.5 was to illustrate the variety of topological structures that are possible for the minimal set  $K_0(f, \infty)$ . Naturally, such examples should be kept simple, but even for the comparatively simply structured

functions  $f_1, \dots, f_5$  in the Examples 6.1 - 6.5, the shape and the connectivity of the minimal set  $K_0(f, \infty)$  has not always been clear at the outset of the analysis.

Naturally, the situation becomes more technical and much more difficult to handle if the function  $f$  becomes more complex, and especially, if it is no longer algebraic. As a consequence, the set  $E_0$  may no longer be finite. For general functions  $f$  it is very difficult to predict shape and connectivity of the minimal set  $K_0(f, \infty)$ . One way to get some information and a rough idea in this respect is to calculate poles of Padé approximants to the function  $f$ . This, by the way, has been done in the study of the function (1.1) in the Introduction, and the result in Figure 1 should be compared with Figure 7.

A critical task for the numerical calculation of the Jordan arcs  $J_j$ ,  $j \in I$ , in the minimal set  $K_0(f, \infty)$  is the calculation of the zeros in the quadratic differential (5.6) in Theorem 9. For this purpose we have developed a numerical procedure, which has been used in the analysis of the Examples 6.3 - 6.5. More details about this topic will be given in Subsection 8.3, further blow.

## 7. A Local Criterion and Geometric Estimates

The  $S$ -property (symmetry property), which has already been introduced and considered in Subsections 5.1, will again take central stage in the first three subsections. We start with a definition of this property that characterises the whole domain, and will then show that it is a local condition for the minimality (2.1) in Definition 2. As a somewhat surprising result in Subsection 7.3, we shall formulate a theorem in which it is proved that the  $S$ -property is also sufficient for the global minimality (2.1) in Definition 2. In the fourth subsections several inclusion relations for the minimal set  $K_0(f, \infty)$  are presented that can be helpful in many practical situations.

**7.1. A General Definition of the  $S$ -Property.** In Theorem 7 of Subsection 5.1 the  $S$ -property (5.1) appears as an important characteristic of the extremal domain  $D_0(f, \infty)$  and its complementary minimal set  $K_0(f, \infty)$ . In the present subsection we define the  $S$ -property for arbitrary admissible domains  $D \in \mathcal{D}(f, \infty)$ . We start with an auxiliary definition.

**DEFINITION 8.** *An admissible domain  $D \in \mathcal{D}(f, \infty)$  for Problem  $(f, \infty)$  is called elementarily maximal if for every point  $z \in \partial D$  one of the following two assertions holds true.*

- (i) *There exists at least one meromorphic continuation of the function  $f$  out of the domain  $D$  that has a non-polar singularity at  $z$ .*
- (ii) *There exist at least two meromorphic continuations of the function  $f$  out of  $D$  that lead to two non-identical function elements at  $z$ .*

It is immediate that if an admissible domain  $D \in \mathcal{D}(f, \infty)$  is not elementarily maximal, then the domain  $D$  can be enlarged in a straight forward way without leaving the class  $\mathcal{D}(f, \infty)$  of admissible domains. Hence, the elementarily maximal domains are the maximal elements in  $\mathcal{D}(f, \infty)$  with respect to ordering by inclusion. We formulate this statement as a proposition.

**PROPOSITION 4.** *The elementarily maximal domains of Definition 8 are the maximal elements in  $\mathcal{D}(f, \infty)$  with respect to ordering by inclusion.*

From the Structure Theorem 4 in Subsection 4.1, we easily deduce that the extremal domain  $D_0(f, \infty)$  is elementarily maximal, but of course, there exist many other maximal elements in  $\mathcal{D}(f, \infty)$ . Often it is helpful, and in most situations also possible, to assume without loss of generality that an arbitrarily chosen admissible domain  $D \in \mathcal{D}(f, \infty)$  is elementarily maximal.

After these preliminaries we come to the definition of the  $S$ -property of a domain.

**DEFINITION 9.** *We say that an admissible domain  $D \in \mathcal{D}(f, \infty)$  possesses the  $S$ -property (symmetry property) with respect to Problem  $(f, \infty)$  if its complement  $K = \overline{\mathbb{C}} \setminus D$  is of the form*

$$K = E_0 \cup E_1 \cup \bigcup_{j \in I} J_j \quad (7.1)$$

and

- (i) *assertion (i) of Definition 8 holds true for every  $z \in \partial E_0$ ,*
- (ii) *assertion (ii) of Definition 8 holds true for every  $z \in K \setminus E_0$ ,*
- (iii) *all  $J_j$ ,  $j \in I$ , are open, analytic Jordan arcs,*
- (iv) *the set  $E_1 \subset K \setminus E_0$  is discrete in  $\mathbb{C} \setminus E_0$ , each point  $z \in E_1$  is the end point of at least three different arcs of  $\{J_j\}_{j \in I}$ , and*
- (v) *if  $I \neq \emptyset$ , then we have*

$$\frac{\partial}{\partial n_+} g_D(z, \infty) = \frac{\partial}{\partial n_-} g_D(z, \infty) \quad \text{for all } z \in J_j, j \in I \quad (7.2)$$

*with  $\partial/\partial n_+$  and  $\partial/\partial n_-$  denoting the normal derivatives to both sides of the arcs  $J_j$ ,  $j \in I$ . By  $g_D(\cdot, \infty)$  we denote the Green function in  $D$ .*

If  $I \neq \emptyset$ , then it is immediate that  $\text{cap}(\partial D) > 0$ , and consequently the Green function  $g_D(z, \infty)$  exists in this case in a proper way (see Subsection 11.3, further below). From identity (7.2) one can deduce that the Jordan arcs  $J_j$ ,  $j \in I$ , are analytic. Hence, the analyticity assumed in assertion (iii) of Definition 9 is implicitly also contained in assertion (v).

Because of the two assertions (i) and (ii) in Definition 9, a domain  $D \in \mathcal{D}(f, \infty)$  with the  $S$ -property is also elementarily maximal in the sense of Definition 8.

With Definition 9 and the Structure Theorem 4, we can rephrase Theorem 7 in Subsection 5.1 as follows: The extremal domain  $D_0(f, \infty)$  possesses the  $S$ -property. In Subsection 7.3, below, we shall see that also the reversed conclusion holds true, i.e., if an admissible domain  $D \in \mathcal{D}(f, \infty)$  possesses the  $S$ -property, then it is identical with the extremal domain  $D_0(f, \infty)$  of Definition 2.

## 7.2. A Local Extremality Condition.

In the present subsection we show that Hadarmard's boundary variation formula for the Green function implies that the  $S$ -property of Definition 9 is a local condition for the minimality of  $\text{cap}(\overline{\mathbb{C}} \setminus D)$ , i.e.,  $\text{cap}(\overline{\mathbb{C}} \setminus D)$  assumes a (local) minimum under local variations of the boundary of an admissible domain  $D \in \mathcal{D}(f, \infty)$  that  $D$  possesses the  $S$ -property.

We start with the introduction of some notations that are needed for the setup of the boundary variation for Hadamard's variation formula (for a very readable introduction to this topic we recommend the appendix of [4]). Let  $D \subset \overline{\mathbb{C}}$  be a domain with  $\infty \in D$ , assume that in  $\partial D$  there exists a smooth, open Jordan arc



$\gamma \subset \partial D$ , and assume further that the domain  $D$  lies only on one side of  $\gamma$ . By  $n(v) \in \mathbb{T}$ , we denote the normal vector on  $\gamma$  at the point  $v \in \gamma$  that shows into  $D$ . Let  $v_0 \in \gamma$  be fixed,  $\varphi \geq 0$  a smooth function defined on  $\gamma$  with compact support. We assume that the support of  $\varphi$  is small and that it contains the point  $v_0 \in \gamma$  in its interior, and choose  $\varepsilon \in \mathbb{R}$  with  $|\varepsilon| > 0$  small.

With these definitions we introduce a variation  $\tilde{D}$  of the domain  $D$  by moving each boundary point  $v \in \gamma$  along the vector  $\varepsilon\varphi(v)n(v)$ . If  $|\varepsilon| > 0$  is sufficiently small, then the new domain  $\tilde{D}$  is well defined. Hadamard's variation formula for the Green function  $g_D(z, w)$  under this type of variation of the domain  $D$  says that

$$g_{\tilde{D}}(z, \infty) - g_D(z, \infty) = \frac{\varepsilon}{2\pi} \int_{\gamma} \frac{\partial}{\partial n} g_D(v, \infty) \frac{\partial}{\partial n} g_D(v, z) \varphi(v) ds_v + O(\varepsilon^2) \quad (7.3)$$

for  $|\varepsilon| \rightarrow 0$ , where  $\partial/\partial n$  denotes the normal derivative and  $ds$  the line element on  $\gamma$ . The Landau symbol  $O(\cdot)$  holds uniformly for  $z$  varying on a compact subset of  $\tilde{D} \cap D$ .

From (7.3) and the connection between the logarithmic capacity and the Green function (cf. Lemma 32 in Subsection 11.3, further below), we then get

$$\frac{\text{cap}(\overline{\mathbb{C}} \setminus \tilde{D})}{\text{cap}(\overline{\mathbb{C}} \setminus D)} - 1 = \frac{\varepsilon}{2\pi} \int_{\gamma} \left( \frac{\partial}{\partial n} g_D(v, \infty) \right)^2 \varphi(v) ds_v + O(\varepsilon^2) \quad (7.4)$$

for  $|\varepsilon| \rightarrow 0$ , which shows that Hadamard's variation formula (7.3) gives us an explicit expression for the first order variation of  $\text{cap}(\overline{\mathbb{C}} \setminus D)$  under local variations of a smooth piece  $\gamma$  of the boundary  $\partial D$ .

Let us now assume that the domain  $D \subset \overline{\mathbb{C}}$  contains a smooth, open Jordan arc  $J \subset \partial D$  with the property that on both sides of  $J$  there are only points of  $D$ , i.e.,  $J$  is a cut in some larger domain. As before, by  $n(v) \in \mathbb{T}$  we denote the normal vector to  $J$  at a point  $v \in J$ , and assume that all normal vectors  $n(v) \in \mathbb{T}$ ,  $v \in J$ , show towards the same side of  $J$ . Again, by  $\varphi \geq 0$  we denote a smooth function on  $J$  with compact support, and assume that the support is contained in a neighborhood of  $v_0 \in J$ . The parameter  $\varepsilon \in \mathbb{R}$  with  $|\varepsilon| > 0$  plays the same role as before.

With the definitions just introduced, we first define a variation  $J_\varepsilon$  of the arc  $J$ . The new arc  $J_\varepsilon$  results from moving each point  $v \in J$  along the vector  $\varepsilon\varphi(v)n(v)$ . For  $|\varepsilon| > 0$  sufficiently small,  $J_\varepsilon$  is well defined and again a smooth Jordan arc. The variation  $D_\varepsilon$  of the domain  $D$  is then defined by replacing the arc  $J$  by  $J_\varepsilon$ . This type of variation changes the boundary  $\partial D$  only locally, but the changes take place in two subregions of  $D$ . The two pieces of the boundary  $\partial D$  that correspond to the arc  $J$  are moved in opposite directions. Because of this variation at two places in  $D$  in opposite directions, we deduce from (7.4) that

$$\frac{\text{cap}(\overline{\mathbb{C}} \setminus D_\varepsilon)}{\text{cap}(\overline{\mathbb{C}} \setminus D)} - 1 = \frac{\varepsilon}{2\pi} \int_J \left[ \left( \frac{\partial}{\partial n_+} g_D(v, \infty) \right)^2 - \left( \frac{\partial}{\partial n_-} g_D(v, \infty) \right)^2 \right] \varphi(v) ds_v + O(\varepsilon^2) \quad (7.5)$$

for  $|\varepsilon| \rightarrow 0$ , where  $\partial/\partial n_+$  and  $\partial/\partial n_-$  denote the normal derivatives to both sides of  $J$ . From (7.5) and the fact that the support of the function  $\varphi$  can be chosen

as small as we want, we can conclude rather immediately that the symmetry (7.2) in Definition 9 of the  $S$ -property is equivalent to the vanishing of the first order variation of  $\text{cap}(\overline{\mathbb{C}} \setminus D)$ . It follows immediately from assertion (ii) in Definition 9 and the local character of the variation of the arc  $J \subset \partial D$  that the resulting variational domain  $D_\varepsilon$  of the original domain  $D$  belongs again to  $\mathcal{D}(f, \infty)$  if  $|\varepsilon| > 0$  is small. The conclusion of our discussion is formulated in the next theorem.

**THEOREM 10.** *Let the complement  $K = \overline{\mathbb{C}} \setminus D$  of an admissible domain  $D \in \mathcal{D}(f, \infty)$  be of the form (7.1) with two sets  $E_0, E_1$ , and the family of arcs  $\{J_j\}$ ,  $j \in I$ , that satisfy the assertions (i) - (iv) in Definition 9. Then the symmetry condition (7.2) holds for every  $z \in J_j$ ,  $j \in I$ , if, and only if, the first order variations of  $\text{cap}(\overline{\mathbb{C}} \setminus D)$  vanish for all local variations of these arcs  $J_j$  done as just described, i.e., if we have*

$$\lim_{|\varepsilon| \rightarrow \infty} \frac{1}{\varepsilon} (\text{cap}(\overline{\mathbb{C}} \setminus D_\varepsilon) - \text{cap}(\overline{\mathbb{C}} \setminus D)) = 0 \quad (7.6)$$

for all such variations.

**7.3.  $S$ -Property and Uniqueness.** In the light of Theorem 10, the result of Theorem 7 can no longer surprise since we now know that the symmetry property (5.1) in Theorem 7 is a necessary condition for the minimality (2.1) in Definition 2. The interesting and perhaps somewhat surprising result in the present section is the next theorem, in which the last conclusion is reversed; it is shown that the  $S$ -property is also sufficient for the minimality (2.1) in Definition 2.

**THEOREM 11.** *If an admissible domain  $D \in \mathcal{D}(f, \infty)$  possesses the  $S$ -property in the sense of Definition 9, then  $D$  is identical with the extremal domain  $D_0(f, \infty)$  of Definition 2.*

Since we know from Theorem 2 that the extremal domain  $D_0(f, \infty)$  is unique, we can deduce a uniqueness result for the extremal domain  $D_0(f, \infty)$  from the  $S$ -property as a corollary to Theorem 11.

**COROLLARY 1.** *The  $S$ -property of an admissible domain  $D \in \mathcal{D}(f, \infty)$  determines uniquely the extremal domain  $D_0(f, \infty)$  of Problem  $(f, \infty)$ .*

The interpretation of the  $S$ -property as a local condition for the minimality (2.1) in Definition 2 is interesting in itself, but it is also interesting for several applications in rational approximation. Hadarmard's variation formula (7.4), on the other hand, is not very helpful as a tool for proofs of the two important Theorems 2 and 4 since it requires the knowledge of smoothness of the arcs  $J_j$ ,  $j \in I$ , in the boundary  $\partial D$ . However, this property is known only when most of the groundwork for the proofs has already been done.

**7.4. Geometric Estimates.** The minimal set  $K_0(f, \infty)$  of Problem  $(f, \infty)$  is in general not convex. The rather trivial Example 6.1 is perhaps the only case, where we have convexity. However, convexity can give rough, and sometimes also quite helpful, geometric estimates for the minimal set  $K_0(f, \infty)$ . Some results in this direction are contained in the next theorem.

**THEOREM 12.** *Let  $K_0(f, \infty)$  be the minimal set for Problem  $(f, \infty)$ , and let further  $E_0 \subset K_0(f, \infty)$  be the compact set that has been introduced in the Structure Theorem 4, i.e.,  $\partial E_0$  contains all non-polar singularities of the function  $f$  that can be reached by meromorphic continuations of the function  $f$  out of the extremal domain  $D_0(f, \infty)$ .*

(i) *For any convex compact set  $K \subset \mathbb{C}$  with the property that the function  $f$  has a single-valued meromorphic continuation throughout  $\overline{\mathbb{C}} \setminus K$ , we have*

$$K_0(f, \infty) \subset K. \quad (7.7)$$

(ii) *Let  $\text{Co}(E_0)$  denote the convex hull of  $E_0$ . Then we have*

$$K_0(f, \infty) \subset \text{Co}(E_0). \quad (7.8)$$

(iii) *Let  $K \subset \mathbb{C}$  be a convex compact set,  $E \subset \mathbb{C} \setminus K$  a set of capacity zero that is closed in  $\mathbb{C} \setminus K$ , and assume that the function  $f$  has a single-valued meromorphic continuation throughout  $\overline{\mathbb{C}} \setminus (K \cup E)$ . Then we have*

$$K_0(f, \infty) \subset K \cup E. \quad (7.9)$$

(iv) *There uniquely exist two sets  $K_{\min} \subset \mathbb{C}$  and  $E_{\min} \subset \mathbb{C} \setminus K_{\min}$  with the same properties as assumed in assertion (iii) for the pairs of sets  $\{K, E\}$  such that these sets are minimal with respect to inclusion among all pairs  $\{K, E\}$  that satisfy the assumptions of assertion (iii), and we have*

$$K_0(f, \infty) \subset K_{\min} \cup E_{\min}. \quad (7.10)$$

(v) *Let  $\text{Ex}(K_{\min})$  denote the set of extreme points of the convex set  $K_{\min}$  from assertion (iv). Then we have*

$$\text{Ex}(K_{\min}) \cup E_{\min} \subset E_0. \quad (7.11)$$

## 8. Geometrically Defined Extremality Problems

Extremality problems are a classical topic in geometric function theory, and among the different versions that are studied there we also find the kind of problems that are concerned with sets of minimal capacity. In the present section our interest concentrates on extremality problems that are defined purely by geometrical conditions since these problems have strong similarities with Problem  $(f, \infty)$ . But there also exist significant differences, which, of course, are the interesting aspects for our discussion.

In order to make this discussion more concrete, and also for later use in proofs, further below, we formulate two classical problems of the geometrical type. The first one is presented in two versions.

### 8.1. Two Classical Problems.

**PROBLEM 1.** *(Chebotarev's Problem) Let finitely many points  $a_1, \dots, a_n \in \mathbb{C}$  be given. Find a continuum  $K \subset \mathbb{C}$  with the property that*

$$a_j \in K \quad \text{for } j = 1, \dots, n, \quad (8.1)$$

*and further that the logarithmic capacity  $\text{cap}(K)$  is minimal among all continua  $K \subset \mathbb{C}$  that satisfy (8.1).*

Problem 1 can be refined in a way that brings it closer to situations that could be observed in the Examples 6.3, 6.4, and 6.5 in Section 6.

**PROBLEM 2.** *Let  $m$  sets  $A_i \subset \mathbb{C}$ ,  $i = 1, \dots, m$ , of finitely many points  $a_{ij} \in A_i$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, m$ , be given. Find  $m$  continua  $K_1, \dots, K_m \subset \mathbb{C}$  with the property that*

$$a_{ij} \in K_i \quad \text{for } i = 1, \dots, m, \quad j = 1, \dots, n_i, \quad (8.2)$$

*and further that the logarithmic capacity  $\text{cap}(K_1 \cup \dots \cup K_m)$  is minimal among all continua  $K_1, \dots, K_m \subset \mathbb{C}$  that satisfy (8.2).*

It is evident that Problem 2 has many similarities to the Problems  $(f, \infty)$  in the Examples 6.1 - 6.5 in Section 6. However, these examples also illustrate some of the essential differences. Especially, there is the question about 'active' versus 'inactive' branch points and also the question about the connectivity of the minimal set  $K_0(f, \infty)$ . Such questions don't exist for the classical problems, since there they are part of the setup of the problem. In Problem  $(f, \infty)$  it is in general not possible to have answers to such questions in advance; the answers are part of the solution and not part of the definition as in Problem 1 and 2.

The functions  $f$  in the Examples 6.1 - 6.5 are rather simple and transparent representatives for the functions possible in Problem  $(f, \infty)$ . In the case of a more complex analytic function  $f$ , the minimal set  $K_0(f, \infty)$  can be very complicated.

From a certain point of view, the two Problems 1 and 2 can be seen as special cases of Problems  $(f, \infty)$ , one has only to choose the function  $f$  in an appropriate way. We exemplify this argument for Problem 2. Let  $f_1$  be defined as

$$f_1(z) := \sum_{i=1}^m \prod_{j=1}^{n_i} \left[ 1 - \frac{a_{ij}}{z} \right]^{1/n_i}, \quad (8.3)$$

then it is immediate that the minimal set  $K_0(f_1, \infty)$  from Theorem 2 is the unique solution of Problem 2.

As a second example for a purely geometrically defined extremality problem we consider the following one:

**PROBLEM 3.** *Let two disjoint, finite sets of points  $a_1, \dots, a_n \in \overline{\mathbb{C}}$  and  $b_1, \dots, b_m \in \overline{\mathbb{C}}$  be given. Find two continua  $K, V \subset \overline{\mathbb{C}}$  with the property that*

$$a_j \in K \quad \text{for } j = 1, \dots, n, \quad b_i \in V \quad \text{for } i = 1, \dots, m, \quad (8.4)$$

*and further that the condenser capacity  $\text{cap}(K, V)$  is minimal among all pairs of continua  $K, V \subset \overline{\mathbb{C}}$  that satisfy (8.4).*

For a definition of the condenser capacity we refer to [27] Chapter II.5. or [1]. Problem 3 has been included here because of two reasons: its solution will be used as an important element in one of the proofs further below, and secondly, it is perhaps the simplest example of its kind with non-unique solutions. In this respect, it underlines the importance and relevance of the uniqueness part in Theorem 2. More about this second aspect follows in the next subsection.

**8.2. Some Methodological and Historic Remarks.** Problem 1 has apparently been mentioned for the first time in a letter by Chebotarev to G. Pólya (see [23]). The existence and uniqueness of a solution for this problem has been proved already shortly afterwards in 1930 by H. Grötzsch [7] with his famous strip method. In [7] one can also find a description of the analytic arcs in the minimal set by quadratic differentials, although the presentation has been done in a different language. In about the same time of [7], M.A. Lavrentiev has formulated and studied Problem 1 in [16] and [17] in an equivalent but somewhat different setting.

A comprehensive review of methods and results relevant for the Problems 1, 2, and 3 can be found in the two long survey papers [13], [14]. We also mention in this respect the textbooks [5] and [24].

In the introduction to the present section it has been mentioned that a wide range of extremality problems has been studied in geometric function theory. There exists a correspondingly broad variety of methods (different types of variational methods, the methods of extremal length, quadratic differentials, etc.) for the analysis of such problems. For our purpose the survey papers [13] and [14] have provided a good coverage of the relevant literature.

In our proofs we shall need only properties of the solution of a special case of Problem 3 (see Definition 18 in Subsection 10.1.1, further below). In this problem the two sets  $A = \{a_1, \dots, a_n\}$  and  $B = \{b_1, \dots, b_n\}$  consist of points which are reflections of each other on the unit circle  $\partial\mathbb{D}$ , i.e., we assume that  $b_j = 1/\bar{a}_j$  for  $j = 1, \dots, n$ . Under this assumption, Problem 3 can be seen as a hyperbolic version of Problem 1. Indeed, the set  $A$  of the given  $n$  points is assumed to be contained in  $\mathbb{D}$  and the logarithmic capacity  $\text{cap}(K)$  in Problem 1 is replaced by the hyperbolic capacity of  $K \subset \mathbb{D}$  (see Subsection 10.1.1, further below).

Our last topic in the present subsection is concerned with the possibility of non-unique solutions to Problem 3. We start with some remarks about Teichmüller's problem, which practically is the most special situation of Problem 3. If in Problem 3 both sets  $A$  and  $B$  consist of only 2 points, then with the help of a Möbius transformation one can show that without loss of generality 3 of the 4 points can be chosen in a standardized way, which usually is done so that  $A = \{-1, 1\}$  and  $B = \{b, \infty\}$  with  $b$  being an arbitrary point in  $\mathbb{C} \setminus \{-1, 1\}$ . Under these assumptions, the minimal condenser capacity  $\text{cap}(K, V)$  of Problem 3 depends only on single complex variable  $b$ . The minimality problem in this special form has been suggested by O. Teichmüller in [41], and it carries today his name. Its solution and the study of its properties has attracted some research interest (cf., [13], Chapter 5.2, for a survey); we mention here only the very recent publication [9], where a numerical method for an efficient calculation of  $\text{cap}(K, V)$  in dependence of  $b \in \mathbb{C} \setminus \{-1, 1\}$  has been developed and studied.

For our discussion, the cases with  $b \in (-1, 1)$  are of special interest, since Teichmüller's problem has non-unique solutions exactly for the parameter values  $b \in (-1, 1)$ . We consider the symmetric case  $b = 0$ .

If in Problem 3, we choose  $n = m = 2$ ,  $\{a_1, a_2\} = \{-1, 1\}$ , and  $\{b_1, b_2\} = \{0, \infty\}$ , then it is not too difficult to verify by symmetry considerations that there exist at least two different solutions  $(K, V)$ . The first one is given by  $K := \{e^{it} \mid \pi \leq t \leq 2\pi\}$  and  $V := \{z \mid 0 \leq \text{Im}(z) \leq \infty, \text{Re}(z) = 0\}$ , while the second one is its symmetric counterpart  $\tilde{K} := \{e^{it} \mid 0 \leq t \leq \pi\}$  and  $\tilde{V} := \{z \mid -\infty \leq \text{Im}(z) \leq 0$ ,

$\operatorname{Im}(z) = 0$  }. This counterexample to uniqueness underlines that the uniqueness part of Theorems 2 cannot be trivial.

The proof of uniqueness of the solution to Problem  $(f, \infty)$  is contained in Subsection 9.3, and it has demanded some new ideas and concepts. A review of the uniqueness question for the general case of Problem 3 is contained in Chapter 5.4 of [13].

**8.3. The Numerical Calculation of the Set  $K_0(f, \infty)$ .** From Theorem 4 we have a general knowledge of the structure of the minimal set  $K_0(f, \infty)$ , and we know that there uniquely exist two compact sets  $E_0$ ,  $E_1$ , and a family of Jordan arcs  $J_j$ ,  $j \in I$ , which are trajectories of a certain quadratic differential, and the union of these objects forms the set  $K_0(f, \infty)$  in (4.1) of Theorem 4. In each concrete case of a function  $f$  that is not as simple as that in Example 6.1, the determination of  $E_0$ ,  $E_1$ , and  $J_j$ ,  $j \in I$ , is a difficult and tricky task, and there is no general method at hand that can be applied in all situations.

The situation is different in the more special case of Theorem 9, where we have a rational quadratic differential  $q(z)dz^2$ , which can be used for the calculation of the Jordan arcs  $J_j$ ,  $j \in I$ . In this more special situation, only two critical tasks have to be done: The first one consists in finding the set of branch points of the function  $f$  in Problem  $(f, \infty)$  that play an active role in the determination of the set  $K_0(f, \infty)$ ; part of this first task is also the determination of the connectivity of the set  $K_0(f, \infty)$ . The second critical task is the calculation of the zeros in the quadratic differential (5.6) in Theorem 9. This second task appears in a similar form if one wants to solve Problem 2, and therefore it has found already earlier attention in the literature. Some results in this direction have been reviewed in the discussion at the end of Example 6.3.

In the analysis of the Examples 6.2 - 6.5 in Section 6, the second task has been solved with the help of a numerical procedure that has been developed in an ad-hoc manner by the author. Details of the procedure will be published elsewhere.

## 9. Proofs I

In the present section we prove Theorem 2 together with the accompanying Propositions 1, 2, and Theorem 3. Thus, we are primarily dealing with a proof of the unique existence of a solution to the Problems  $(f, \infty)$ . Like in Theorem 2, we assume throughout the section that the function  $f$  is meromorphic in the neighborhood of infinity.

**9.1. Meromorphic Continuations Along Arcs.** The continuation of a function element along a given arc  $\gamma$  is basic for any technique of meromorphic continuations. In the present subsection we introduce special sets of arcs and curves, and define on them a homotopy relation that is adapted to our special needs in later proofs. Toward the end of the subsection in Proposition 5, we prove a characterization of the domains in  $\mathcal{D}(f, \infty)$  in terms of these newly introduced tools, i.e., a characterization of admissible domains for Problem  $(f, \infty)$ .

As a general notational convention, we denote the impression of a curve or an arc by the same symbol as use for the curve or the arc itself.

**DEFINITION 10.** *By  $\Gamma = \Gamma(f, \infty)$  we denote the set of all Jordan curves  $\gamma$  with the following two properties:*

- (i) We have  $\infty \in \gamma$ .
- (ii) There exists a point  $z \in \gamma \setminus \{\infty\}$ , called separation point of  $\gamma$ , such that the curve  $\gamma$  is broken down into the two closed partial arcs  $\gamma^-$  and  $\gamma^+$  connecting the two points  $z$  and  $\infty$ . The function  $f$  is assumed to possess meromorphic continuations along each of the two arcs, and these two arcs are not identical, i.e., we have  $\gamma = \gamma^+ - \gamma^-$  and  $\gamma^+ \cap \gamma^- = \{z, \infty\}$ . ('Closed' means here the arc contains its end points).

We assume that each Jordan curve  $\gamma \in \Gamma$  has a parametrization of the form

$$\gamma : [-1, 1] \longrightarrow \overline{\mathbb{C}} \quad (9.1)$$

with  $\gamma(-1) = \gamma(1) = \infty$  and  $\gamma(0) = z$ .

From (9.1), we have the parametrization

$$\gamma^+ : [1, 0] \longrightarrow \overline{\mathbb{C}}, \quad \gamma^- : [-1, 0] \longrightarrow \overline{\mathbb{C}} \quad (9.2)$$

for the two partial arcs  $\gamma^-$  and  $\gamma^+$ .

Whether a Jordan curve  $\gamma$  with  $\infty \in \gamma$  belongs to  $\Gamma$  depends on the function  $f$ . A necessary and sufficient condition can be formulated as follows: We have  $\gamma \in \Gamma$  if, and only if, the two meromorphic continuations of  $f$  that start at  $\infty$  and follow  $\gamma$  in the two different directions cover the whole curve  $\gamma$ . We emphasize that the two continuations may hit non-polar singularities somewhere on the curve  $\gamma$ , but this is only allowed to happen after the separation point has already been passed.

Throughout the present section we assume that the separation point  $z = z_\gamma \in \gamma \in \Gamma$  is chosen in an appropriate way, and we give details only if necessary.

In the next definition the set  $\Gamma$  is divided into two subclasses.

**DEFINITION 11.** A Jordan curve  $\gamma \in \Gamma = \Gamma(f, \infty)$  with partial arcs  $\gamma^-$  and  $\gamma^+$  belongs to the subclass  $\Gamma_0 = \Gamma_0(f, \infty) \subset \Gamma$  if the meromorphic continuations of the function  $f$  along the two arcs  $\gamma^-$  and  $\gamma^+$  lead to the same function element at the separation point  $z$  of  $\gamma$ . If, on the other hand, these continuations lead to two different function elements at  $z$ , then the curve  $\gamma$  belongs to the subclass  $\Gamma_1 = \Gamma_1(f, \infty) \subset \Gamma$ .

It is immediate that the two subsets  $\Gamma_0$  and  $\Gamma_1$  are disjoint, and we have  $\Gamma = \Gamma_0 \cup \Gamma_1$ .

On the set  $\Gamma$  we define a homotopy relation that fits our special needs. Two elements  $\gamma_0, \gamma_1 \in \Gamma$  are considered to be homotopic if the two pairs  $\{\gamma_0^-, \gamma_1^-\}$  and  $\{\gamma_0^+, \gamma_1^+\}$  of partial arcs are homotopic in the usual sense, and if in addition property (ii) in Definition 10 is carried over from one to the other Jordan curve  $\gamma_0$  and  $\gamma_1$  in a continuous manner. More formally, we have the next definition.

**DEFINITION 12.** Two Jordan curves  $\gamma_0, \gamma_1 \in \Gamma$  with partial arcs  $\gamma_j^\pm$ ,  $j = 0, 1$ , and separation points  $z_j$ ,  $j = 0, 1$ , are called homotopic (written  $\sim$ ) if there exists a continuous function  $h : [-1, 1] \times [0, 1] \longrightarrow \overline{\mathbb{C}}$  with the two following two properties:

- (i) For  $j = 0, 1$ , we have

$$\gamma_j(t) = h(t, j), \quad t \in [-1, 1]. \quad (9.3)$$

- (ii) For each  $s \in (0, 1)$  a Jordan curve  $\gamma_s$  is defined by

$$\gamma_s := h(\cdot, s) : [-1, 1] \longrightarrow \overline{\mathbb{C}}, \quad (9.4)$$

and each  $\gamma_s$  belongs to  $\Gamma$  with separation point  $\gamma_s(0)$ .

The equivalence class of  $\gamma \in \Gamma$  with respect to the homotopy relation  $\sim$  is denoted by  $[\gamma]$ .

LEMMA 1. *The splitting of the set  $\Gamma$  into the two subclasses  $\Gamma_0$  and  $\Gamma_1$  of Definition 11 is compatible with the homotopy relation of Definition 12.*

PROOF. The conclusion of the lemma is immediate.  $\square$

The ring domain  $R \subset \overline{\mathbb{C}}$  and the continuum  $V \subset \mathbb{C}$  in the next lemma will be used at several places in the sequel. We say that  $R$  is a ring domain in  $\overline{\mathbb{C}}$  if  $\overline{\mathbb{C}} \setminus R$  consists of two components.

LEMMA 2. *For any  $\gamma_0 \in \Gamma = \Gamma(f, \infty)$  there exists a ring domain  $R \subset \overline{\mathbb{C}}$  with  $\gamma_0 \subset R$ , for which the following five assertions hold true:*

- (i) *The Jordan curve  $\gamma_0$  separates the two components  $A_1$  and  $A_2$  of  $\overline{\mathbb{C}} \setminus R$ .*
- (ii) *Any Jordan curve  $\gamma \subset R$  with  $\infty \in \gamma$  that separates the two components  $A_1$  and  $A_2$  of  $\overline{\mathbb{C}} \setminus R$  belongs to  $\Gamma$ .*
- (iii) *Any  $\gamma \in \Gamma$  with  $\gamma \subset R$  belong to  $\gamma \in [\gamma_0]$ , i.e., we have  $\gamma \sim \gamma_0$  in the sense of Definition 12.*
- (iv) *If a Jordan curve  $\gamma \subset R$  separates the two components  $A_1$  and  $A_2$  of  $\overline{\mathbb{C}} \setminus R$ , then any Jordan curve  $\tilde{\gamma} \subset R$  with  $\infty \in \tilde{\gamma}$ , which is homotopic to  $\gamma$  in  $R$  (in the usual sense), belongs to  $\Gamma$ .*
- (v) *If  $\gamma_0 \in \Gamma_1 = \Gamma_1(f, \infty)$ , then every admissible compact set  $K \in \mathcal{K}(f, \infty)$  contains a continuum  $V \subset \mathbb{C}$  that cross-sects  $R$ , i.e., we have*

$$V \cap A_j \neq \emptyset \quad \text{for } j = 1, 2 \quad (9.5)$$

*with  $A_1$  and  $A_2$  the two components of  $\overline{\mathbb{C}} \setminus R$ . The set  $\mathcal{K}(f, \infty)$  has been introduced in Definition 1.*

PROOF. Let  $U^-$  and  $U^+$  be two open and simply connected neighborhoods of the partial arcs  $\gamma_0^-$  and  $\gamma_0^+$  of  $\gamma_0$ , and let the function  $f$  possess meromorphic continuations throughout  $U^-$  and  $U^+$ . Let further  $z_0 = \gamma_0(0)$  denote the separation point of  $\gamma_0$ . By using  $\varepsilon$ -neighborhoods of  $\gamma_0^-$  and  $\gamma_0^+$ , one can easily show that there exists a ring domain  $R \subset \overline{\mathbb{C}}$  and an open disk  $U_0$  with  $z_0$  as its centre such that

$$\overline{U}_0 \subset U^- \cap U^+, \quad \gamma_0 \subset R \subset U^- \cup U^+, \quad (9.6)$$

$$R \cap \partial U_0 \text{ has exactly two components, and} \quad (9.7)$$

$$R \setminus \overline{U}_0 \text{ is a simply connected domain.} \quad (9.8)$$

Assertion (i) immediately follows from the construction of the ring domain  $R$  if the  $\varepsilon$ -neighborhoods of  $\gamma_0^-$  and  $\gamma_0^+$  are chosen sufficiently narrow.

Assertion (ii) follows from the following two facts: (a) any Jordan curve  $\gamma$  in  $R$  that separates the two components  $A_1$  and  $A_2$  is homotopic to  $\gamma_0$  in the usual sense, and (b)  $\gamma$  will intersect with  $U_0$  because of (9.7) and (9.8). From the last assertion, it follows that we can choose a separation point  $z$  for  $\gamma$  anywhere in  $\gamma \cap U_0$ .

The assertions (iii) and (iv) are obvious completions of assertion (ii), and they follow rather immediately from the construction of  $R$  in (9.6), (9.7), and (9.8).

For the proof of assertion (v) we assume that  $K$  is an arbitrary element of  $\mathcal{K}(f, \infty)$ , i.e.,  $\overline{\mathbb{C}} \setminus K$  is an admissible domain for Problem  $(f, \infty)$  as introduced in Definition 1, and further we assume that  $\gamma_0 \in \Gamma_1$ .



We considered the open set  $R \setminus K$ . From  $\gamma_0 \in \Gamma_1$  and assertion (iv) it follows that

$$\gamma \cap K \neq \emptyset \quad (9.9)$$

for all Jordan curves  $\gamma \subset R$  that separate  $A_1$  from  $A_2$ . Indeed, if (9.10) were false for some Jordan curve  $\gamma$ , then this curve could be modified near infinity in  $R \setminus K$  into a Jordan curve  $\tilde{\gamma} \subset R \setminus K$  that is homotopic to  $\gamma$  in  $R$  and  $\infty \in \tilde{\gamma}$ . From assertion (iv) we then know that  $\tilde{\gamma} \in \Gamma$ . Since  $R \setminus K \subset \overline{\mathbb{C}} \setminus K \in \mathcal{D}(f, \infty)$ , we know from Definition 1 that the function  $f$  has a single-valued meromorphic continuation along the whole curve  $\tilde{\gamma}$ , which implies that  $\tilde{\gamma} \in \Gamma_0$ . On the other hand, from the assumption  $\gamma_0 \in \Gamma_1$  we deduce with assertion (iii) that also  $\gamma_1 \in \Gamma_1$ . This last contradiction proves (9.9).

Assertion (v) then follows from (9.9) and the next Lemma 3. The lemma is of independent interest for several applications at other places in the sequel.  $\square$

LEMMA 3. *Let  $R \subset \overline{\mathbb{C}}$  be a ring domain,  $A_1$  and  $A_2$  the two components of  $\overline{\mathbb{C}} \setminus R$ , and let  $K \subset \mathbb{C}$  be a compact set. There exists a continuum  $V \subset K$  with*

$$V \cap A_j \neq \emptyset \quad \text{for } j = 1, 2 \quad (9.10)$$

*if and only if*

$$\gamma \cap K \neq \emptyset \quad (9.11)$$

*for every Jordan curve  $\gamma \subset R$  that separates  $A_1$  from  $A_2$ .*

PROOF. Let us first assume that there exists a Jordan curve  $\gamma$  with the given properties for which (9.11) is false, and let  $O_1$  and  $O_2$  be the interior and the exterior domain of  $\gamma$ . Then for any continuum  $V \subset K$  satisfying (9.10) we would have the contradiction that  $V \subset O_1 \cup O_2$  and  $V \cap O_j \neq \emptyset$  for  $j = 1, 2$ .

Next, we assume that (9.11) holds true, and set  $B_0 := \overline{R} \cap K$ . Let  $C_j \subset B_0$ ,  $j \in I$ , be the family of all components of  $B_0$  that are disjoint from at least one of the two sets  $A_1$  or  $A_2$ . The set  $I$  is denumerable, we assume  $I \subset \mathbb{N}$ , and define

$$B_n := \overline{B_0 \setminus \bigcup_{j \in I, j \leq n} C_j} \quad \text{for } n = 1, 2, \dots \quad (9.12)$$

The assumption of (9.11) implies that also

$$\gamma \cap B_n \neq \emptyset \quad \text{for } n > 0 \quad (9.13)$$

and for every Jordan curve  $\gamma \subset R$  that separates  $A_1$  from  $A_2$ . Indeed, if there would exist an exceptional Jordan curve  $\gamma$ , then  $\gamma$  could be modified into a Jordan curve  $\tilde{\gamma} \subset R \setminus B_0$  that is homotopic to  $\gamma$  in  $R$ , which then would contradict (9.11).

From (9.13) we deduce that

$$B_\infty := \bigcap_{n \in \mathbb{N}} B_n \neq \emptyset. \quad (9.14)$$

The set  $B_\infty$  contains only components that intersect simultaneously both sets  $A_1$  and  $A_2$ , which proves the existence of a continuum  $V \subset B_\infty \subset K$  satisfying (9.10).  $\square$

The following proposition has been the main reason and motivation for the introduction of the sets  $\Gamma$ ,  $\Gamma_0$ , and  $\Gamma_1$  of Jordan curves in the Definitions 10 and 11.

PROPOSITION 5. Let  $\Gamma = \Gamma(f, \infty)$ ,  $\Gamma_0, \Gamma_1 \subset \Gamma$  be the sets of Jordan curves introduced in the two Definitions 10 and 11, and let  $\mathcal{D}(f, \infty)$  be the set of admissible domains for Problem  $(f, \infty)$  introduced in Definition 1.

A domain  $D \subset \overline{\mathbb{C}}$  with  $\infty \in D$  belongs to  $\mathcal{D}(f, \infty)$  if, and only if, the following two assertions hold true:

- (i) The function  $f$  has a meromorphic continuation along each closed Jordan arc  $\gamma$  in  $D$  that starts at  $\infty$ .
- (ii) For each Jordan curve  $\gamma \in \Gamma_1$  we have  $\gamma \cap (\overline{\mathbb{C}} \setminus D) \neq \emptyset$ .

PROOF. Assertion (i) ensures that the function  $f$  has a meromorphic continuation to each point of the domain  $D$ , and assertion (ii) guarantees that these continuations are single-valued. Hence, the two assertions (i) and (ii) imply  $D \in \mathcal{D}(f, \infty)$ .

The other direction of the proof follows also rather immediately from the two Definitions 1 and 11. If  $D \in \mathcal{D}(f, \infty)$ , then clearly assertions (i) holds true; and if there would exist  $\gamma_1 \in \Gamma_1$  with  $\gamma_1 \subset D$ , then this would contradict the assumption in Definitions 1 that the meromorphic continuation of the function  $f$  in  $D$  is single-valued.  $\square$

**9.2. The Existence of a Domain in  $\mathcal{D}_0(f, \infty)$ .** In Definition 2, the set of all admissible domains with a complement of minimal capacity has been denoted by  $\mathcal{D}_0(f, \infty)$ . In the present subsection we prove that  $\mathcal{D}_0(f, \infty)$  is not empty.

PROPOSITION 6. We have  $\mathcal{D}_0(f, \infty) \neq \emptyset$ .

The basic structure of the proof of Proposition 6 is simple and straightforward: A minimizing sequence of admissible compact sets  $K_n \in \mathcal{K}(f, \infty)$ ,  $n \in \mathbb{N}$ , is chosen in such a way that in the limit the minimality condition (2.1) in Definition 2 is satisfied. The transition to the limit situation is done in the frame work of potential theory. It is shown that after some plausible corrections the resulting domain is admissible for Problem  $(f, \infty)$ . However, in the practical realization a number of technical hurdles have to be overcome; the whole proof is broken down in several consecutive steps, which are presented as lemmas.

In a first step, we deal with the very special situation that we have the value zero in the minimality (2.1) of Definition 2.

LEMMA 4. If in (2.1) of Definition 2 we have

$$\inf_{K \in \mathcal{K}(f, \infty)} \text{cap}(K) = 0, \quad (9.15)$$

then the subclass  $\Gamma_1(f, \infty)$  of Jordan curves introduced in Definition 11 is empty.

PROOF. For an indirect proof we assume  $\Gamma_1 = \Gamma_1(f, \infty) \neq \emptyset$ . Let  $\gamma_0$  be an element of  $\Gamma_1$ , and let further  $R \subset \overline{\mathbb{C}}$  be a ring domain with  $\gamma_0 \subset R$  as introduced in Lemma 2. From assertion (iv) of Lemma 2 it follows that for every admissible compact set  $K \in \mathcal{K}(f, \infty)$  there exists a continuum  $V \subset K$  that intersects  $R$ , i.e., we have

$$V \cap A_j \neq \emptyset \quad \text{for } j = 1, 2 \quad (9.16)$$

and  $A_1, A_2$  the two components of  $\overline{\mathbb{C}} \setminus R$ . From the lower estimate (11.4) for the capacity given in Lemma 20, further below, we then conclude that

$$\text{cap}(K) \geq \text{diam}(V)/4 \geq \text{dist}(A_1, A_2)/4. \quad (9.17)$$

Since the right-hand side of (9.17) is independent of  $V$  and the choice of  $K$ , the estimate (9.17) contradicts (9.15). Thus, we have proved that  $\Gamma_1 = \emptyset$ .  $\square$

In Lemma 4 a special case of Proposition 2 has been addressed, and we have the following corollary.

**COROLLARY 2.** *If condition (9.15) is satisfied, then all meromorphic continuations of the function  $f$  are single-valued, and consequently, the extremal domain  $D_0 = D_0(f, \infty)$  of Definition 2 is the Weierstrass domain  $W_f \subset \overline{\mathbb{C}}$  for meromorphic continuation of the function  $f$  starting at  $\infty$ .*

**PROOF.** It follows immediately from Definition 11 that  $\Gamma_1 = \emptyset$  is equivalent to the single-valuedness of all meromorphic continuations of  $f$  in  $\overline{\mathbb{C}}$ , and consequently we have  $D_0(f, \infty) = W_f$ .  $\square$

Thanks to Lemma 4, we can now assume without loss of generality for the remainder of the present subsection that

$$\inf_{K \in \mathcal{K}(f, \infty)} \text{cap}(K) =: c_0 > 0. \quad (9.18)$$

We select a sequence of admissible compact sets  $K_n \in \mathcal{K}(f, \infty)$ ,  $n \in \mathbb{N}$ , such that

$$\lim_{n \rightarrow \infty} \text{cap}(K_n) = c_0. \quad (9.19)$$

**LEMMA 5.** *There exists  $r > 0$  such that we can assume without loss of generality that the sequence  $\{K_n\}$  in (9.19) satisfies*

$$K_n \subset \{|z| \leq r\} \quad \text{for all } n \in \mathbb{N}. \quad (9.20)$$

**PROOF.** Let  $r > 1$  be such that  $f$  is meromorphic in  $\{|z| > r - 1\}$ . For any admissible compact set  $K \in \mathcal{K}(f, \infty)$ , we denote by  $\tilde{K}$  the radial projection of  $K$  onto the disk  $\{|z| \leq r\}$  as defined in (11.6) of Subsection 11.1, further below. It is not difficult to verify that because of  $\overline{\mathbb{C}} \setminus K \in \mathcal{D}(f, \infty)$  we also have  $\tilde{D} := \overline{\mathbb{C}} \setminus \tilde{K} \in \mathcal{D}(f, \infty)$ . One has only to check the conditions in Definition 1.

From Lemma 22 in Subsection 11.1, it then follows that  $\text{cap}(\tilde{K}) \leq \text{cap}(K)$ . Hence, any compact set  $K_n$  in (9.19), which does not satisfy (9.20), can be replaced by  $\tilde{K}_n$ , and because of (9.18), the limit (9.19) remains unchanged under such modifications.  $\square$

In the sequel we assume that the inclusions (9.20) hold true for all compact sets  $K_n \in \mathcal{K}(f, \infty)$ ,  $n \in \mathbb{N}$ , in (9.19).

Let  $\omega_n$  be the equilibrium distribution of the compact set  $K_n$ ,  $n \in \mathbb{N}$ , and let further  $g_n = g_{D_n}(\cdot, \infty)$  be the Green function in the domain  $D_n$  (for definitions of  $\omega_n$  and  $g_n$  see Section 11, further below). As explained in Subsection 11.4, there always exists an infinite subsequence  $N \subset \mathbb{N}$  such that the weak\* limit

$$\omega_n \xrightarrow{*} \omega_0 \quad \text{as } n \rightarrow \infty, n \in N. \quad (9.21)$$

exists. Since inclusion (9.20) has been assumed to hold true for the sequence  $\{K_n\}$ , we have

$$\text{supp}(\omega_0) \subset \{|z| \leq r\}, \quad (9.22)$$

and  $\omega_0$  is a probability measures.

Using representation (11.45) of Lemma 32 for the Green function  $g_n$  we have

$$g_n = -p(\omega_n; \cdot) - \log \text{cap}(K_n) \quad (9.23)$$

with  $p(\omega_n; \cdot)$  denotes the logarithmic potential of  $\omega_n$ , which has formerly been defined in Subsection 11.2, further below. From limit (9.21) and the Lower Envelope Theorem 16 of potential theory (cf. Subsection 11.2, further below) we then conclude that

$$\limsup_N g_n \leq g_0 := -p(\omega_0; \cdot) - \log(c_0) \quad (9.24)$$

with the constant  $c_0$  introduced in (9.18). In (9.24), equality holds quasi everywhere in  $\mathbb{C}$  (for a definition of "quasi everywhere" see Definition 21, further below). It follows from (9.21) and (9.22) that outside of  $\{|z| \leq r\}$  we have a proper limit in (9.24) instead of the limes superior, and equality holds there instead of the inequality stated in (9.24). In  $\{|z| > r\}$ , the limit in (9.24) holds locally uniformly.

DEFINITION 13. *We define*

$$\tilde{K}_0 := \overline{\{z \in \mathbb{C} \mid g_0(z) = 0\}}, \quad (9.25)$$

and  $\tilde{D}_0 := \overline{\mathbb{C}} \setminus \tilde{K}_0$ .

The two sets  $\tilde{D}_0$  and  $\tilde{K}_0$  will become building blocks for extremal domains and minimal sets of Problem  $(f, \infty)$ , but several modifications and special considerations have to be made before the construction can be finished.

We note that the two sets  $\tilde{D}_0$  and  $\tilde{K}_0$ , like the measure  $\omega_0$  and the function  $g_0$ , depend on the subsequence  $N \subset \mathbb{N}$  used in the limit (9.21).

LEMMA 6. *We have*

$$\text{cap}(\tilde{K}_0) \leq c_0 = \inf_{D \in \mathcal{D}(f, \infty)} \text{cap}(\overline{\mathbb{C}} \setminus D), \quad (9.26)$$

$\tilde{K}_0 \subset \{|z| \leq r\}$ , and  $\tilde{D}_0$  is a domain.

PROOF. The function  $g_0$  introduced in (9.24) is subharmonic in  $\mathbb{C}$ , which implies that the set  $\tilde{D}_0$  introduced in Definition 13 is a domain.

The inclusion  $\tilde{K}_0 \subset \{|z| \leq r\}$  is an immediate consequence of (9.22).

It remains to prove (9.26). From (9.24) it follows that  $g_0 \geq 0$  everywhere in  $\mathbb{C}$ . Since the logarithmic potential of a finite measure is continuous quasi everywhere in  $\mathbb{C}$  (cf. the introductory paragraphs of Subsection 11.2, further below), we conclude from (9.25) that

$$g_0(z) = 0 \quad \text{for quasi every } z \in \tilde{K}_0. \quad (9.27)$$

We can assume without loss of generality that  $\text{cap}(\tilde{K}_0) > 0$  since otherwise (9.26) is trivially true. From Lemma 25 in Subsection 11.2 we then know that the equilibrium distribution  $\tilde{\omega}_0$  on  $\tilde{K}_0$  is of finite energy. Hence, we can use the Principle of Domination from Theorem 17 in Subsection 11.2 for a comparison of the function  $g_0 = -p(\omega_0; \cdot) - \log(c_0)$  from (9.24) with the Green function  $g_{\overline{\mathbb{C}} \setminus \tilde{K}_0}(\cdot, \infty) = -p(\tilde{\omega}_0; \cdot) - \log \text{cap}(\tilde{K}_0)$ . In the last equation, we have used representation (11.45) from Lemma 32 in Subsection 11.3. With the Principle of Domination we deduce from (9.27) that

$$g_{\overline{\mathbb{C}} \setminus \tilde{K}_0}(z, \infty) \geq g_0(z) \quad \text{for all } z \in \overline{\mathbb{C}}. \quad (9.28)$$

Comparing both side in (9.28) near infinity yields the inequality

$$\log \text{cap}(\tilde{K}_0) \leq \log c_0, \quad (9.29)$$

which proves (9.26).  $\square$

It will turn out in (9.46) and (9.47), further below, that in (9.26) we always have equality. Because of (9.28), this means that we have  $\omega_0 = \tilde{\omega}_0$ , and therefore the measure  $\omega_0$  has no mass points outside of  $\tilde{K}_0$ .

In the next lemma, we see that  $\overline{\mathbb{C}} \setminus \tilde{K}_0$  is indeed an important building block for an admissible domain with a minimal capacity, i.e., an element of  $\mathcal{D}_0(f, \infty)$ . The result should be seen in relation to Proposition 5.

LEMMA 7. *We have  $\gamma \cap \tilde{K}_0 \neq \emptyset$  for every Jordan curve  $\gamma \in \Gamma_1$ .*

PROOF. Let  $\gamma_0$  be an arbitrary element of  $\Gamma_1$  with  $\Gamma_1 = \Gamma_1(f, \infty)$  introduced in Definition 11, and let further  $R \subset \overline{\mathbb{C}}$  be a ring domain with  $\gamma_0 \subset R$  as introduced in Lemma 2. Since  $\gamma_0 \in \Gamma_1$ , we know from assertion (iv) in Lemma 2 that for every  $n \in \mathbb{N}$  there exists a continuum  $V_n \subset K_n$  which cross-sects the ring domain  $R$ , i.e., we have

$$V_n \cap A_j \neq \emptyset \quad \text{for } j = 1, 2 \quad (9.30)$$

with  $A_1$  and  $A_2$  the two components of  $\overline{\mathbb{C}} \setminus R$ .

Using Lemma 42 from Subsection 11.4 together with the assumptions (9.19), (9.20), and (9.21), we conclude from (9.30) that there exists a continuum  $V \subset \{|z| \leq r\}$  that satisfy (11.102) and (11.103) in Lemma 42, i.e., we have

$$V \cap A_j \neq \emptyset \quad \text{for } j = 1, 2, \quad \text{and} \quad (9.31)$$

$$g_0(z) = 0 \quad \text{for } z \in V. \quad (9.32)$$

From (9.25) and (9.32), we then conclude that  $V \subset \tilde{K}_0$ . Because of (9.31), we also know from Lemma 3 that  $\gamma_0 \cap V \neq \emptyset$ , and consequently, we have shown that  $\gamma_0 \cap \tilde{K}_0 \neq \emptyset$ .  $\square$

In the proof of Lemma 7, Lemma 42 from Subsection 11.4, further below, has played a key role. The lemma will be essential at several other places in the sequel, and since its proof in Subsection 11.4 is based on Carathéodory's Theorem about kernel convergence, we can say that the proof of the last lemma and also that of the other results is essentially based on Carathéodory's fundamental Theorem. This theorem gives also importance to the use of the continua  $V$  that have already appeared in the two Lemmas 2 and 3.

As a corollary of Lemma 7, we deduce that any meromorphic continuation of the function  $f$  in the domain  $\tilde{D}_0$  is single-valued. Thus, one of the two conditions in Proposition 5 for a characterization of an admissible domain is satisfied by  $\tilde{D}_0$ . What we still have not investigated is the question whether the function  $f$  can be meromorphically continued to every point of  $\tilde{D}_0$ , or how large the set of exceptional points in  $\tilde{D}_0$  can be if such a continuation is not possible. We start the investigation with the following definition.

DEFINITION 14. *Let  $\tilde{E}_0 \subset \tilde{D}_0$  be the set of all points  $z \in \tilde{D}_0$  that satisfy the following two conditions:*

- (i) *There exists a Jordan arc  $\gamma$  in  $\tilde{D}_0$  with initial point  $\infty$  and end point  $z$ , and the function  $f$  has a meromorphic continuation along  $\gamma \setminus \{z\}$ .*
- (ii) *At the point  $z$ , the meromorphic continuation of  $f$  along  $\gamma$  has a non-polar singularity.*

The next lemma is an immediate consequence of Lemma 7.

LEMMA 8. *Let  $z_0 \in \tilde{D}_0$ , and let  $\gamma_1, \gamma_2$  be two Jordan arcs that both satisfy condition (i) in Definition 14 with  $z_0$  as end point. Then condition (ii) of Definition 14 is either simultaneously satisfied, or simultaneously not satisfied by the two arcs  $\gamma_1$  and  $\gamma_2$ .*

PROOF. Let us assume that  $\gamma_1$  and  $\gamma_2$  are two Jordan arcs, which both satisfy condition (i) in Definition 14 with  $z_0$  as end point, let  $\gamma_1$  satisfy also condition (ii), but  $\gamma_2$  not. Then after some modifications, if necessary, the composition  $\gamma := \gamma_1 - \gamma_2$  is a Jordan curve in  $\tilde{D}_0$ . A separation point in the sense of Definition 10 can be chosen on  $\gamma_1$  in the neighborhood of  $z_0$ , and we then have  $\gamma \in \Gamma$  with  $\Gamma = \Gamma(f, \infty)$  introduced in Definition 10. It follows from the assumptions made with respect to  $\gamma_1$  and  $\gamma_2$  together with Definition 10 that  $\gamma \in \Gamma_1$ , but this would contradict Lemma 7.  $\square$

The next lemma is a preparation of a proof of the result that  $\text{cap}(\tilde{E}_0) = 0$  for the set  $\tilde{E}_0$  that has been introduced in Definition 14. Not only the formulation but also the proof this lemma is rather technical.

LEMMA 9. *Let  $D_1$  be a simply connected and bounded domain with  $\overline{D}_1 \subset \tilde{D}_0$ . Then there exists  $n_1 \in \mathbb{N}$  such that*

$$\tilde{E}_0 \cap \overline{D}_1 \subset K_n \quad \text{for all } n \geq n_1, n \in N \quad (9.33)$$

*with  $N \subset \mathbb{N}$  the subsequence used in the limit (9.21).*

PROOF. With the assumptions made with respect to the domain  $D_1$  it is rather immediate that there exists a ring domain  $R \subset \tilde{D}_0$  such that for one of the two components  $A_1$  and  $A_2$  of  $\overline{\mathbb{C}} \setminus R$ , say  $A_1$ , we have

$$\overline{D}_1 \subset A_1 \subset \tilde{D}_0 \quad \text{and} \quad \infty \in R. \quad (9.34)$$

In addition to the domain  $D_1$ , we define the domain  $D_2 := A_1 \cup R \subset \tilde{D}_0$ .

In a first step of the proof we show that there exists  $n_1 \in \mathbb{N}$  such that for each  $n \geq n_1, n \in N$ , there exists at least one Jordan curve

$$\gamma \in \Gamma_0 = \Gamma_0(f, \infty) \quad \text{with} \quad \gamma \subset R \setminus K_n. \quad (9.35)$$

The proof of (9.35) will be given indirectly. For a negation of (9.35) we consider the following two cases a and b:

Case a: There exists an infinite subsequence  $N_1 \subset N$  such that for each  $n \in N_1$  and each Jordan curve  $\gamma \subset R$  that separates  $A_1$  from  $A_2$  we have

$$\gamma \cap K_n \neq \emptyset. \quad (9.36)$$

Case b: There exists an infinite subsequence  $N_2 \subset N$  such that for each  $n \in N_2$  there exists at least one Jordan curve

$$\gamma \in \Gamma_1 = \Gamma_1(f, \infty) \quad \text{with} \quad \gamma \subset R \setminus K_n. \quad (9.37)$$

The sets  $\Gamma_0$  and  $\Gamma_1$  have been introduced in Definition 11.

If we have disproved Case a, then it follows from assertion (ii) of Lemma 2 that there exists  $\gamma \in \Gamma$  with  $\gamma \subset R \setminus K_n$  for every  $n \in N$  sufficiently large, and consequently, either (9.35) or (9.37) holds true for the particular Jordan curve  $\gamma$ . If also Case b is disproved, then it follows from assertion (iii) in Lemma 2 that for every  $n \in N$  sufficiently large there exists a Jordan curve  $\gamma$  which satisfies (9.35), and we have accomplished the first step of the proof.

In order to disprove Case a, we observe that from Lemma 3 together with (9.36), it follows that for each  $n \in N_1$  there exists a continuum  $V_n \subset K_n$  with

$$V_n \cap A_j \neq \emptyset \quad \text{for } j = 1, 2. \quad (9.38)$$

As in the relations (9.31) and (9.32) in the proof of Lemma 7, we deduce from (9.38) with the help of Lemma 42 in Subsection 11.4, further below, that there exists a continuum  $V \subset \tilde{K}_0$  with  $V \cap R \neq \emptyset$ , but the existence of  $V$  contradicts the assumption  $R \subset \tilde{D}_0$ . Hence, Case a is disproved.

In order to disprove Case b, we observe that from the existence of the Jordan curve  $\gamma$  in (9.37) and assertion (iv) in Lemma 2 it follows that for each  $n \in N_2$  there exists a continuum  $V_n \subset K_n$  that satisfies (9.38). With the same arguments as just used after (9.38), we come to the same conclusion that there exists  $V \subset \tilde{K}_0$  with  $V \cap R \neq \emptyset$ , which again contradicts  $R \subset \tilde{D}_0$ , and thus, also Case b is disproved.

As already said before, with a disproof of the two Cases a and b, we have shown that there exists  $n_1 \in \mathbb{N}$  such that for every  $n \geq n_1$ ,  $n \in N$  there exists a Jordan curve  $\gamma$  which satisfies (9.35).

In a second step, we prove indirectly the relations (9.33) for  $n \geq n_1$ ,  $n \in N$ . Let us assume that  $(\tilde{E}_0 \cap \overline{D}_1) \setminus K_{n_0} \neq \emptyset$  for a certain  $n_0 \geq n_1$ ,  $n_0 \in N$ . Then there exists a point

$$z_0 \in (\tilde{E}_0 \cap \overline{D}_1) \setminus K_{n_0}. \quad (9.39)$$

Let in accordance with Definition 14  $\gamma_1 \subset \tilde{D}_0$  be a Jordan arc with initial point  $\infty$  and end point  $z_0$  such that the two conditions (i) and (ii) of Definition 14 are satisfied.

Since the function  $f$  is single-valued and meromorphic in  $D_{n_0} = \overline{\mathbb{C}} \setminus K_{n_0} \in \mathcal{D}(f, \infty)$ , there exists a Jordan arc  $\tilde{\gamma}_2 \subset D_{n_0}$  with initial point  $\infty$  and end point  $z_0$  and  $f$  is meromorphic along  $\tilde{\gamma}_2$ .

We know from (9.35) that in  $R \setminus K_{n_0} = R \cap D_{n_0}$  there exists a Jordan curve  $\gamma_0$  with  $\infty \in \gamma_0$ , and along  $\gamma_0$  the function  $f$  has a single-valued meromorphic continuation. Hence, we can modify the arc  $\tilde{\gamma}_2$  in such a way that the modified Jordan arc  $\gamma_2$  coincides with  $\tilde{\gamma}_2$  after its last contact with  $\gamma_0$ , but the whole Jordan arc  $\gamma_2$  is contained in  $D_{n_0} \cap D_2 \subset D_{n_0} \cap \tilde{D}_0$ , and it connects  $\infty$  with  $z_0$ . Clearly, the new Jordan arc  $\gamma_2$  satisfies condition (i) of Definition 14, but it does not satisfy condition (ii). Hence, the two Jordan arcs  $\gamma_1$  and  $\gamma_2$  contradict Lemma 8. This contradiction disproves the existence of  $z_0$  in (9.39), and completes the proof of lemma.  $\square$

The proof of Lemma 9 has been very technical since in its background logic we were confronted with the possibility that in each admissible compact set  $K_n$ ,  $n \in N$ , different non-polar singularities of the function  $f$  may be 'active' or 'inactive'. Illustrations for this phenomenon have been given in the examples of Section 6. In the situation of Lemma 9, it has turned out that with the selection of the

subsequence  $N \subset \mathbb{N}$  in (9.21) all relevant choices in this respect have been fixed by the set  $\tilde{K}_0$ .

LEMMA 10. *We have*

$$\text{cap}(\tilde{E}_0) = 0 \quad (9.40)$$

for the set  $\tilde{E}_0$  introduced in Definition 14.

PROOF. For an indirect proof we assume that

$$\text{cap}(\tilde{E}_0) > 0. \quad (9.41)$$

Using an exhaustion of the domain  $\tilde{D}_0 \cap \{|z| \leq r\}$  by overlapping closed and simply connected domains, one can show because of (9.41) that there exists a simply connected domain  $D_1$  with  $\overline{D}_1 \subset \tilde{D}_0 \cap \{|z| \leq r\}$  such that

$$\text{cap}(\tilde{E}_0 \cap \overline{D}_1) > 0. \quad (9.42)$$

The constant  $r$  is the same as that in Lemma 5. From (9.42) and Lemma 9, we know that there exists  $n_1 \in \mathbb{N}$  such that (9.34) holds true. Using Lemma 41 from Subsection 11.4, further below, we deduce from (9.34) together with the assumptions (9.19) and (9.21) that we have

$$g_0(z) = 0 \quad \text{for quasi every } z \in \tilde{E}_0 \cap \overline{D}_1 \quad (9.43)$$

with  $g_0$  the function introduced in (9.24). From (9.42), (9.43), and (9.25) in Definition 13, it then follows that  $\tilde{E}_0 \cap \tilde{K}_0 \neq \emptyset$ , but this contradicts Definition 14, and thus, the lemma is proved.  $\square$

With the two Definitions 13 and 14, the Lemmas 6, 7, 10, and Proposition 5 we are prepared to prove Proposition 6 and close the present subsection.

**Proof of Proposition 6.** In a first step, we deal with the special case that (9.15) holds true. We then know from Lemma 4 and its Corollary 2 that the extremal domain  $D_0 = D_0(f, \infty) \in \mathcal{D}_0(f, \infty)$  of Definition 2 exists and is identical with the Weierstrass domain  $W_f \subset \mathbb{C}$  for meromorphic continuations of the function  $f$  starting at  $\infty$ .

In the second step, we assume that the inequality in (9.18) is satisfied, and define

$$D_0 := \tilde{D}_0 \setminus \tilde{E}_0, \quad (9.44)$$

and show that  $D_0 \in \mathcal{D}_0(f, \infty)$ .

The set  $\tilde{E}_0$  is identical to its polynomial-convex hull  $\hat{E}_0$ . Indeed, from Lemma 7 and from Lemma 23 in Subsection 11.1, further below, we deduce that

$$\text{cap}(\hat{E}_0) = \text{cap}(\tilde{E}_0) = 0. \quad (9.45)$$

From (9.45) it follows that  $\hat{E}_0$  can have no inner points, and consequently, we have  $\hat{E}_0 = \tilde{E}_0$ . This identity together with (9.44) and Lemma 6 implies that  $D_0$  is a domain.

From Lemma 7 we know that condition (ii) in Proposition 5 is satisfied. From (9.44) and Definition 14 it follows that also condition (i) in Proposition 5 is satisfied. It therefore follows from Proposition 5 that  $D_0$  is an admissible domain for Problem  $(f, \infty)$ , i.e.,  $D_0 \in \mathcal{D}(f, \infty)$ .



Since the capacity of a capacitable set does not change its value if a set of capacity zero is added or subtracted (cf. Lemma 21 in Subsection 11.1, further below), we deduce from the two Lemmas 6 and 10 that

$$\text{cap}(\overline{\mathbb{C}} \setminus D_0) = \text{cap}(\overline{\mathbb{C}} \setminus \tilde{D}_0) = \text{cap}(\tilde{K}_0) \leq c_0 \quad (9.46)$$

with the constant  $c_0$  introduced in (9.18). From the minimality (9.18) and the fact that  $D_0 \in \mathcal{D}(f, \infty)$ , we conclude that in (9.46) a proper inequality is not possible. Hence, we have proved that

$$\text{cap}(\overline{\mathbb{C}} \setminus D_0) = \inf_{D \in \mathcal{D}(f, \infty)} \text{cap}(\overline{\mathbb{C}} \setminus D), \quad (9.47)$$

which implies that  $D_0 \in \mathcal{D}_0(f, \infty)$ , and the proof of Proposition 6 completed. ■

**9.3. Uniqueness up to a Set of Capacity Zero.** In the present subsection we prove that all admissible domains in  $\mathcal{D}_0(f, \infty)$  differ only in a set of capacity zero. In Section 2, this result has already been stated as the first part of Proposition 1, and there the sets  $\mathcal{D}_0(f, \infty)$  and  $\mathcal{K}_0(f, \infty)$  have also been introduced in Definition 2. We formulate the result here as a proposition, which then will be proved at the end of the subsection after several auxiliary results have been formulated and proved.

**PROPOSITION 7.** *Sets in  $\mathcal{K}_0(f, \infty)$  differ at most in a subset of capacity zero.*

A key role in the proof of Proposition 7 is played by a number of special sets that are introduced in Definition 15 below. Especially the construction of the compact set  $K_0$  in (9.58) can be seen as a type of convex combination, which will become more clear in Subsection 9.5, further below.

We start with the formal set-up for an indirect proof of Proposition 7 and assume contrary to the assertion of the proposition that there exist at least two minimal compact sets  $K_1, K_2 \in \mathcal{K}_0(f, \infty)$  that differ in a set of positive capacity, i.e., we assume

$$\text{cap}((K_1 \setminus K_2) \cup (K_2 \setminus K_1)) > 0 \quad \text{for } K_1, K_2 \in \mathcal{K}_0(f, \infty). \quad (9.48)$$

The corresponding admissible domains are defined as

$$D_j := \overline{\mathbb{C}} \setminus K_j \in \mathcal{D}_0(f, \infty), \quad j = 1, 2. \quad (9.49)$$

Since we have assumed  $K_1, K_2 \in \mathcal{K}_0(f, \infty)$ , we know that the minimality (2.1) in Definition 2 holds for both sets, i.e., we have

$$\text{cap}(K_1) = \text{cap}(K_2) = \inf_{K \in \mathcal{K}(f, \infty)} \text{cap}(K) = c_0 \quad (9.50)$$

with  $c_0$  the same constant as that introduced in (9.18). The two Green functions  $g_{D_j}(\cdot, \infty)$  in the two domains  $D_j$ ,  $j = 1, 2$ , are denoted by

$$g_j := g_{D_j}(\cdot, \infty), \quad j = 1, 2. \quad (9.51)$$

From Lemma 33 in Subsection 11.3, further below, we know that assumption (9.48) is equivalent to the assertion that the two Green functions  $g_1$  and  $g_2$  are not identical. From the harmonicity of  $g_1 - g_2$  in  $D_1 \cap D_2$ , it then follows that

$$g_1(z) \neq g_2(z) \quad \text{for almost all } z \in D_1 \cap D_2, \quad (9.52)$$

and equality holds in  $D_1 \cap D_2$  on piece-wise analytic arcs. These arcs are part of the set  $S_0$  that is formally defined in (9.53) in the next definition. All sets introduced in

Definition 15 will appear in subsequent lemmas that lead to the proof of Proposition 7 at the end of the present subsection.

DEFINITION 15. *Under the assumptions (9.48) and (9.50) and with the same notations as introduced in (9.49) and (9.51), we define the sets  $S_0, K_3, \tilde{K}_0, K_{10}, K_{20}, K_0, D_0 \subset \overline{\mathbb{C}}$  in the following way:*

$$S_0 := \overline{\{z \in \overline{\mathbb{C}} \mid g_1(z) = g_2(z)\}}, \quad (9.53)$$

$$K_3 := \widehat{K_1 \cup K_2}, \quad (9.54)$$

$$\tilde{K}_0 := \widehat{S_0 \cap K_3}, \quad (9.55)$$

$$K_{10} := \{z \in K_1 \setminus \tilde{K}_0 \mid g_1(z) > g_2(z)\}, \quad (9.56)$$

$$K_{20} := \{z \in K_2 \setminus \tilde{K}_0 \mid g_2(z) > g_1(z)\}, \quad (9.57)$$

$$K_0 := \tilde{K}_0 \cup K_{10} \cup K_{20}, \quad (9.58)$$

$$D_0 := \overline{\mathbb{C}} \setminus K_0. \quad (9.59)$$

The polynomial-convex hull of a bounded set  $S \subset \mathbb{C}$  is denoted by  $\hat{S}$  (cf. Definition 22 in Subsection 11.1, further below).

For the proof of Proposition 7 the following strategy will be applied: First, it is proved that the set  $D_0$  introduced in (9.59) is a domain. Then it is shown that  $D_0$  is an admissible domain, i.e.,  $D_0 \in \mathcal{D}(f, \infty)$ . After that in the final step, it is proved that the assumptions (9.48) and (9.50) imply that  $\text{cap}(K_0) < c_0$  for the compact set introduced in (9.58). But such an estimate contradicts the minimality assumed in (9.50). From a methodological point of view the last step is the most interesting and also the most challenging one.

We start with two lemmas in which topological and some potential-theoretic properties of sets from Definition 15 are investigated. The first lemma is of a more preparatory character.

LEMMA 11. *We set*

$$d := g_1 - g_2, \quad (9.60)$$

*and define the two sets*

$$B_+ := \{z \in \overline{\mathbb{C}} \setminus S_0 \mid g_1(z) > g_2(z)\} = \{z \in \overline{\mathbb{C}} \setminus S_0 \mid d(z) > 0\}, \quad (9.61)$$

$$B_- := \{z \in \overline{\mathbb{C}} \setminus S_0 \mid g_1(z) < g_2(z)\} = \{z \in \overline{\mathbb{C}} \setminus S_0 \mid d(z) < 0\}. \quad (9.62)$$

*Both sets are open. The function  $d$  is superharmonic in  $B_+$  and subharmonic in  $B_-$ .*

PROOF. Let  $C \subset \overline{\mathbb{C}}$  be an arbitrary component of  $\overline{\mathbb{C}} \setminus S_0$ . This component is broken down into the two sets

$$C_1 := C \cap B_+ \quad \text{and} \quad C_2 := C \cap B_-. \quad (9.63)$$

Since the function  $d$  is the difference of two Green functions, we know from Lemma 30 in Subsection 11.3, further below, that  $d$  is continuous outside of a measurable set  $A \subset \overline{\mathbb{C}}$  with  $\text{cap}(A) = 0$ . The definitions (9.63) together with the continuity of  $d$  then imply that

$$\partial C_j \cap C \subset A \quad \text{for } j = 1, 2. \quad (9.64)$$

Indeed, if we assume that  $z \in \partial C_1 \cap C$  and  $z \notin A$ , then it follows from the continuity of  $d$  in  $C \setminus A$  that  $d(z) = 0$ , and therefore  $z \in S_0$ , which contradicts the definition of the component  $C$ . For  $j = 2$  the same conclusion holds true.

Next, we show that we can have

$$C_j \setminus A \neq \emptyset \quad (9.65)$$

at most for one of the two possibilities  $j = 1, 2$ . Indeed, it follows from  $\text{cap}(A) = 0$  and from Lemma 24 in Subsection 11.1, further below, that  $C \setminus A$  is connected. If we assume that (9.65) holds for both  $j = 1, 2$ , then it follows from the continuity of  $d$  in  $C \setminus A$  that there exists  $z \in C \setminus A$  with  $d(z) = 0$ . But this would imply that  $z \in S_0$ , which again contradicts the definition of the component  $C$ . We assume without loss of generality that

$$C_2 \subset A \quad \text{and} \quad C_1 \supset C \setminus A. \quad (9.66)$$

Let, as in Definition 25 of Subsection 11.3, further below,  $Rg(K_1)$  denote the set  $K_1$  minus all irregular points of  $K_1$ . From (9.66) it follows that

$$C \cap \overline{Rg(K_1)} = \emptyset. \quad (9.67)$$

Indeed, if there exists  $z \in C \cap \overline{Rg(K_1)}$ , then we know from part (iv) of Lemma 31 in Subsection 11.3, further below, that  $\text{cap}(C \cap K_1) > 0$ , and further with Lemma 30 in Subsection 11.3 we conclude that also  $\text{cap}(C \cap Rg(K_1)) > 0$ .

Since  $g_1(z) = 0$  for all  $z \in Rg(K_1)$ , it follows that  $d(z) \leq 0$  for  $z \in Rg(K_1)$  and therefore that  $d(z) < 0$  for  $z \in C \cap Rg(K_1)$ . With (9.66) this implies that  $C \cap Rg(K_1) \subset A$ , and consequently, we have  $\text{cap}(C \cap Rg(K_1)) = 0$ . Since the last conclusion has led to a contradiction, (9.67) is proved.

From (9.67) and part (iii) of Lemma 31 in Subsection 11.3, further below, we conclude that the function  $d$  is superharmonic in the component  $C$ . Indeed, from the Lemma 31 we know that  $g_1$  is harmonic in  $C$ , and on the other hand,  $-g_2$  is superharmonic in  $C$  (cf. Lemma 34 in Subsection 11.3).

Since the minimum principle is valid for superharmonic functions, we conclude from (9.66) and (9.63) that  $d(z) > 0$  for all  $z \in C$ , and consequently, we have proved

$$C \subset B_+. \quad (9.68)$$

The conclusion (9.68) is conditional on the assumption made in (9.64). The alternative choice in (9.64) would have led to a reversed role for the two subsets  $C_+$  and  $C_-$ , and we would have proved that  $C \subset B_-$ , and further that the function  $d$  is subharmonic in  $C$ .

Putting both possibilities together, we have proved that each component of the open set  $\mathbb{C} \setminus S_0$  is completely contained in one of the two subsets  $B_+$  or  $B_-$ . Hence, we have shown that these two sets are open. Further, it has been shown that the function  $d$  is superharmonic (resp. subharmonic) in  $B_+$  (resp. in  $B_-$ ).  $\square$

LEMMA 12. (i) *The two sets*

$$B_1 := (K_3 \setminus \tilde{K}_0) \cap B_+ \quad \text{and} \quad B_2 := (K_3 \setminus \tilde{K}_0) \cap B_- \quad (9.69)$$

*are disjoint, and they are open in  $K_3$ .*

(ii) We have

$$B_j \setminus K_j = B_j \setminus K_{j0} \quad \text{for } j = 1, 2, \quad \text{and} \quad (9.70)$$

$$\text{cap}(K_{10}) = \text{cap}(K_{20}) = 0. \quad (9.71)$$

(iii) Set  $D_3 := \overline{\mathbb{C}} \setminus K_3$ , then both sets

$$D_3 \cup (B_j \setminus K_j), \quad j = 1, 2, \quad (9.72)$$

are domains.

(vi) The set  $D_0$  from (9.59) is a domain, and we have the decomposition

$$D_0 = D_3 \cup (B_1 \setminus K_1) \cup (B_2 \setminus K_2). \quad (9.73)$$

PROOF. Assertion (i) is an immediate consequence of Lemma 11.

In order to prove assertion (ii), we observe that it follows from the definition of  $K_{10}$  in (9.56) together with (9.61) and (9.69) that  $K_{10} = B_1 \cap K_1$ , which implies (9.70) for  $j = 1$ .

From the introduction of  $K_{10}$  in (9.56) and the definition of irregular points in the Definitions 24 and 25 in Subsection 11.3, further below, it further follows that  $K_{10} \subset \text{Ir}(K_1)$ , and the first part of (9.71) therefore is a consequence of Lemma 30 in Subsection 11.3. Assertion (ii) follows analogously for  $j = 2$ .

Since  $\tilde{K}_0$  is polynomial-convex by definition, each component of  $K_3 \setminus \tilde{K}_0$  is a subset of a component of  $\overline{\mathbb{C}} \setminus S_0$ , which implies that  $D_3 \cup B_j$  is a domain for each  $j = 1, 2$ . Assertion (iii) then follows immediately from the second part of Lemma 24 in Subsection 11.1, further below, together with (9.70) and (9.71).

For a proof of assertion (iv), we observe that

$$\begin{aligned} D_0 &= \overline{\mathbb{C}} \setminus (\tilde{K}_0 \cup K_{10} \cup K_{20}) = (\overline{\mathbb{C}} \setminus K_3) \cup (K_3 \setminus \tilde{K}_0) \setminus (K_{10} \cup K_{20}) \\ &= D_3 \cup (B_1 \setminus K_1) \cup (B_2 \setminus K_2). \end{aligned} \quad (9.74)$$

Indeed, the identities follow from the defining relations of the sets in Definition 15 together with (9.70). From (9.74) and assertion (iii) of the present lemma, it follows that  $D_0$  is a domain.  $\square$

The main result in Lemma 12 is the assertion that the set  $D_0$  is a domain.

LEMMA 13. *The domain  $D_0$  introduced in (9.59) of Definition 15 is admissible for Problem  $(f, \infty)$ , i.e., we have  $D_0 \in \mathcal{D}(f, \infty)$  with  $\mathcal{D}(f, \infty)$  introduced in Definition 1.*

PROOF. The basis of the proof is Proposition 5.

In a first step we prove that assertion (i) of Proposition 5 holds true. Let  $f_j$ ,  $j = 1, 2$ , be the two single-valued meromorphic continuations of the function  $f$  in Problem  $(f, \infty)$  in the two admissible domains  $D_j$ ,  $j = 1, 2$ , in (9.49). In  $D_3$  both functions are identical; but beyond this domain, the situation is different since we have assumed that the two domains  $D_1$  and  $D_2$  are not identical; the set  $K_3$  has been defined in (9.54). Using the two sets from (9.69), we defined a new function  $f_0$  as

$$f_0(z) := \begin{cases} f_1(z) = f_2(z) & \text{for } z \in D_3, \\ f_1(z) & \text{for } z \in B_1 \setminus K_1, \\ f_2(z) & \text{for } z \in B_2 \setminus K_2. \end{cases} \quad (9.75)$$

It follows from the assertions (iii) and (iv) of Lemma 12 that  $f_0$  is a meromorphic continuation of the function  $f$  into  $D_0$ , which shows that assertion (i) of Proposition 5 holds true.

Assertion (ii) in Proposition 5 is a direct consequence of assertions (i) in Lemma 12. Hence, it follows from Proposition 5 that  $D_0 \in \mathcal{D}(f, \infty)$ .  $\square$

We come now to the main part of the analysis in the present subsection, the proof of the inequality  $\text{cap}(K_0) < c_0$  for the compact set  $K_0 = \overline{\mathbb{C}} \setminus D_0$  introduced in (9.58) in Definition 15. In this proof two auxiliary functions  $h_0$  and  $h_1$  will play an important role; they are introduced in the next definition.

DEFINITION 16. *With the sets introduced in Definition 15 and the notation  $g_1$  and  $g_2$  for the Green functions (9.51), we define two functions  $h_0$  and  $h_1$  by*

$$h_0(z) := \begin{cases} \frac{1}{2}(g_1(z) + g_2(z)) & \text{for } z \in \overline{\mathbb{C}} \setminus K_3, \\ \frac{1}{2}|g_1(z) - g_2(z)| & \text{for } z \in K_3 \setminus \text{Int}(\tilde{K}_0), \\ 0 & \text{for } z \in \text{Int}(\tilde{K}_0), \end{cases} \quad (9.76)$$

$$h_1(z) := \begin{cases} \frac{1}{2}|g_1(z) - g_2(z)| & \text{for } z \in \overline{\mathbb{C}} \setminus K_3, \\ \frac{1}{2}(g_1(z) + g_2(z)) & \text{for } z \in K_3 \setminus \text{Int}(\tilde{K}_0), \\ \frac{1}{2}\widehat{(g_1(z) + g_2(z))} & \text{for } z \in \text{Int}(\tilde{K}_0). \end{cases} \quad (9.77)$$

In (9.77),  $\widehat{g_1 + g_2}$  is the solution of the Dirichlet problem in each component  $C$  of the interior  $\text{Int}(\tilde{K}_0)$  of  $\tilde{K}_0$  with  $(g_1 + g_2)|_{\partial\tilde{K}_0}$  as boundary function.

In the next lemma a number of technical details are proved; they will be needed in the subsequent lemma.

LEMMA 14. *If the assumptions (9.48) and (9.50) are satisfied, then with the notations from the Definitions 15, 16, and Lemma 12, the following assertions hold true:*

(i) *There exists a signed measure  $\sigma_0$  of finite energy with*

$$\text{supp}(\sigma_0) \subset (K_1 \cup K_2) \setminus \text{Int}(\tilde{K}_0) \quad (9.78)$$

*such that the function  $h_0$  from (9.77) has the representation*

$$h_0 = g_0(\cdot, \infty) + \int g_0(\cdot, v) d\sigma_0(v), \quad (9.79)$$

*where  $g_0(\cdot, \cdot)$  is the Green function in the domain  $D_0$ . The measure  $\sigma_0$  is carried by the set*

$$\Sigma_0 := (K_1 \cup K_2) \setminus \tilde{K}_0, \quad (9.80)$$

*and we have*

$$\sigma_0 \neq 0. \quad (9.81)$$

(ii) *There exists a signed measure  $\sigma_1$  of finite energy with*

$$\text{supp}(\sigma_1) \subset S_0 \cup K_3 \quad (9.82)$$

*such that the function  $h_1$  from (9.77) has the representation*

$$h_1 = p(\sigma_1; \cdot), \quad (9.83)$$

where  $p(\sigma_1; \cdot)$  denotes the logarithmic potential of the measure  $\sigma_1$  as introduced in (11.13) in Subsection 11.2, further below. We have

$$\sigma_1(\mathbb{C}) = 0. \quad (9.84)$$

(iii) With the notation (9.80), we have

$$h_0(z) = h_1(z) \quad \text{for quasi every } z \in \Sigma_0, \quad (9.85)$$

$$h_1(z) = 0 \quad \text{for quasi every } z \in S_0 \setminus \text{Int}(K_3), \quad (9.86)$$

$$\sigma_0 = -\sigma_1|_{K_3 \setminus \tilde{K}_0}, \quad (9.87)$$

$$\sigma_1|_{\text{Int}(K_3)} \leq 0. \quad (9.88)$$

PROOF. In the first part of the proof we show that the two functions  $h_0$  and  $h_1$  introduced in (9.76) and (9.77) can be represented by potentials with measures of finite energy. This is done by an investigation of a sequence of auxiliary functions.

By  $h_2$  we denote the function

$$h_2 := \frac{1}{2}(g_1 + g_2) = r_2 + p(\sigma_2; \cdot), \quad (9.89)$$

where the last equality is a consequence of (9.51) together with representation (11.45) for Green functions in Lemma 32 in Subsection 11.3, further below. It follows from (9.50) and (9.51) together with Lemma 32 that we have

$$r_2 = -\log(c_0) \quad \text{and} \quad \sigma_2 = -\frac{1}{2}(\omega_1 + \omega_2) \leq 0 \quad (9.90)$$

with  $\omega_j$  the equilibrium distribution on  $K_j$ ,  $j = 1, 2$ .

From Lemma 27 together with Lemma 32 in the two Subsections 11.2 and 11.3, further below, we know that the function  $\frac{1}{2}|g_1 - g_2|$  can also be represented as a potential. We denote this function by  $h_3$ ; and we have

$$h_3 := \frac{1}{2}|g_1 - g_2| = p(\sigma_3; \cdot) \quad (9.91)$$

with  $\sigma_3$  a signed measure of finite energy and

$$\text{supp}(\sigma_3) \subset S_0 \cup K_1 \cup K_2. \quad (9.92)$$

In (9.91), there is no constant because of (9.50).

From Lemma 30 in Subsection 11.3, further below, we know that the two Green functions  $g_1$  and  $g_2$ , and consequently also the two functions  $h_2$  and  $h_3$  are continuous quasi everywhere in  $\mathbb{C}$ . Hence, it follows from (9.53) in Definition 15 and (9.91) that

$$h_3(z) = 0 \quad \text{for quasi every } z \in S_0. \quad (9.93)$$

Because of (9.54) in Definition 15, the two Green functions  $g_1$  and  $g_2$  are harmonic outside of  $K_3$ , and therefore we have equality for every  $z \in S_0 \setminus K_3$  in (9.93) without any exception.

Next, we use the balayage technique (cf. Definition 23 in Section 11.2, further below) for sweeping the masses of the two measures  $\sigma_2$  and  $\sigma_3$  out of the open set  $\text{Int}(\tilde{K}_0)$ . The two resulting balayage measures are denoted by  $\sigma_4$  and  $\sigma_5$ , respectively. From part (i) of Definition 23 of the balayage applied to the measure  $\sigma_2$ , we

get as consequence that the new function  $h_4$  is of the form

$$h_4 := r_2 + p(\sigma_4; \cdot) = \begin{cases} \frac{1}{2}(\widehat{g_1 + g_2}) & \text{in } \text{Int}(\tilde{K}_0), \\ h_2 & \text{quasi everywhere on } \partial\tilde{K}_0, \\ h_2 & \text{in } \mathbb{C} \setminus \tilde{K}_0. \end{cases} \quad (9.94)$$

In (9.94) we have used (9.89). About the measure  $\sigma_4$  we know that

$$\text{supp}(\sigma_4) \subset \text{supp}(\sigma_2) \setminus \text{Int}(\tilde{K}_0), \quad (9.95)$$

$$\sigma_4|_{\mathbb{C} \setminus \tilde{K}_0} = \sigma_2|_{\mathbb{C} \setminus \tilde{K}_0} = -\frac{1}{2}(\omega_1 + \omega_2)|_{\mathbb{C} \setminus \tilde{K}_0}. \quad (9.96)$$

On the other hand, from the balayage of the measure  $\sigma_3$ , it follows that the new function  $h_5$  is of the form

$$h_5 := p(\sigma_5; \cdot) = \begin{cases} 0 & \text{in } \text{Int}(\tilde{K}_0), \\ 0 & \text{quasi everywhere on } \partial\tilde{K}_0, \\ h_3 & \text{in } \mathbb{C} \setminus \tilde{K}_0. \end{cases} \quad (9.97)$$

For the derivation of (9.97) identity (9.91) has been used. The balayage measure  $\sigma_5$  satisfies

$$\text{supp}(\sigma_5) \subset \text{supp}(\sigma_3) \setminus \text{Int}(\tilde{K}_0), \quad (9.98)$$

$$\sigma_5|_{\mathbb{C} \setminus \tilde{K}_0} = \sigma_3|_{\mathbb{C} \setminus \tilde{K}_0}. \quad (9.99)$$

The function  $\frac{1}{2}(\widehat{g_1 + g_2})$  in the first line of (9.94) is the solution of the Dirichlet problem in each component of  $\text{Int}(\tilde{K}_0)$  with boundary function  $h_2|_{\partial\tilde{K}_0}$  (cf. (11.21) in Definition 23 in Section 11.2, further below). Analogously, in the first line of (9.97) the function  $h_5 = 0$  is the solution of the Dirichlet problem in each component of  $\text{Int}(\tilde{K}_0)$  with boundary function  $h_3|_{\partial\tilde{K}_0}$  since from (9.93) we know that  $h_3(z) = 0$  for quasi every  $z \in \partial\tilde{K}_0$ .

The functions  $h_4$  and  $h_5$  and their associated measures  $\sigma_4$  and  $\sigma_5$  are the building blocks for the presentations by potentials for the two functions  $h_0$  and  $h_1$  introduced in Definition 16. The proof of existence of such potentials is the main objectives in the present analysis. In the next step this aim will be achieved by using a method what pasting potentials together, which is described in Lemma 28 of Subsection 11.2, further below.

Because of (9.54) in Definition 15 we have

$$\partial K_3 \subset K_1 \cup K_2 \subset K_3, \quad (9.100)$$

and consequently

$$h_4 = h_5 \quad \text{quasi everywhere on } \partial K_3, \quad (9.101)$$

which together with the representations (9.94) and (9.97) shows that the assumptions of Lemma 28 in Subsection 11.2 are satisfied. The domain  $D$  in Lemma 28 is now  $\text{Int}(K_3)$ .

From (9.76), (9.77), and the technique described in Lemma 28, we deduce that there exists two signed measures  $\tilde{\sigma}_0$  and  $\sigma_1$  of finite energy such that

$$h_0(z) = r_2 + p(\tilde{\sigma}_0; z) \quad \text{for quasi every } z \in \mathbb{C}, \quad (9.102)$$

$$h_1(z) = p(\sigma_1; z) \quad \text{for quasi every } z \in \mathbb{C}. \quad (9.103)$$

Thus, in (9.102) and (9.103) we have representations by potentials for the piecewise defined functions  $h_0$  and  $h_1$ , respectively.

Because of (9.100) and the properties of the functions  $h_4$  and  $h_5$  in the two sets  $\text{Int}(K_3)$  and  $\overline{\mathbb{C}} \setminus \text{Int}(K_3)$ , we have

$$\text{supp}(\tilde{\sigma}_0) \subset (K_1 \cup K_2 \cup \tilde{K}_0) \setminus \text{Int}(\tilde{K}_0), \quad (9.104)$$

$$\text{supp}(\sigma_1) \subset (S_0 \setminus \text{Int}(K_3)) \cup (K_1 \cup K_2 \cup \tilde{K}_0) \setminus \text{Int}(\tilde{K}_0). \quad (9.105)$$

From (11.27) in Lemma 28 in Subsection 11.2 together with (9.77), (9.89), and (9.94), it further follows that

$$\sigma_1|_{\text{Int}(K_3)} = \sigma_4|_{\text{Int}(K_3)} \leq 0. \quad (9.106)$$

The inequality in (9.106) is a consequence of (9.96).

In order to prove a relationship between the two measures  $\tilde{\sigma}_0$  and  $\sigma_1$ , we observe that from (9.76) and (9.77) in Definition 16 it follows that

$$h_0(z) + h_1(z) = \max(g_1(z), g_2(z)) \quad \text{for all } z \in \overline{\mathbb{C}} \setminus \tilde{K}_0. \quad (9.107)$$

From (9.107) and Lemma 11 we deduce that  $h_0 + h_1$  is harmonic in  $\mathbb{C} \setminus S_0$ . Indeed, on the set  $B_+$  introduced in Lemma 11, we have  $h_0 + h_1 = g_1$ . Since we know from Lemma 11 that the function  $d = g_1 - g_2$  is superharmonic in  $B_+$ , we deduce that  $g_1$  is harmonic in  $B_+$ . On the set  $B_-$  in Lemma 11, analogous considerations hold true.

From (9.76) and (9.77) in Definition 16 together with the two representations (9.102), (9.103), and the harmonicity of  $h_0 + h_1$  in  $\mathbb{C} \setminus S_0$ , it follows that

$$\tilde{\sigma}_0|_{\mathbb{C} \setminus S_0} = -\sigma_1|_{\mathbb{C} \setminus S_0}, \quad (9.108)$$

which is the relation between  $\tilde{\sigma}_0$  and  $\sigma_1$  we were looking for.

From (9.58) in Definition 15 and (9.71) in Lemma 12 we know that  $K_0$  and  $\tilde{K}_0$  differ only in a set of capacity zero. Hence, from the defining property (11.43) for Green functions, which has been stated at the beginning of Subsection 11.3, we then conclude that

$$g_0(\cdot, v) := g_{\overline{\mathbb{C}} \setminus K_0}(\cdot, v) \equiv g_{\overline{\mathbb{C}} \setminus \tilde{K}_0}(\cdot, v) \quad \text{for all } v \in D_0. \quad (9.109)$$

In the next step, we investigate the relation between Green function  $g_0 = g_0(\cdot, \infty)$  and the function  $h_0$ . From (9.76) in Definition 16 together with (9.94), (9.93), and (9.55), we conclude that

$$h_0(z) = 0 \quad \text{for quasi every } z \in \tilde{K}_0. \quad (9.110)$$

For the function  $h_0$  we have representation (9.102). We will now show that if we sweep the measure  $\tilde{\sigma}_0$  out of the domain  $\overline{\mathbb{C}} \setminus \tilde{K}_0$  by balayage, we arrive at the Green function  $g_0(\cdot, v)$ . Let  $\hat{\sigma}_0$  be the balayage measure on  $\partial \tilde{K}_0$  resulting from sweeping  $\tilde{\sigma}_0$  out of  $\overline{\mathbb{C}} \setminus \tilde{K}_0$ , then it follows from Definition 23, part (ii), in Subsection



11.2 together with formula (11.51) in Lemma 34 in Subsection 11.3 that

$$\begin{aligned}
r_2 + p(\widehat{\sigma}_0; z) - \int_{\mathbb{C} \setminus \widetilde{K}_0} g_0(v, \infty) d\widetilde{\sigma}_0(v) \\
= r_2 + p(\widetilde{\sigma}_0; z) - \int_{\mathbb{C} \setminus \widetilde{K}_0} g_0(z, v) d\widetilde{\sigma}_0(v) \\
= h_0(z) - \int_{\mathbb{C} \setminus \widetilde{K}_0} g_0(z, v) d\widetilde{\sigma}_0(v) = 0
\end{aligned} \tag{9.111}$$

for quasi every  $z \in \widetilde{K}_0$ . In (9.111) the last equality follows from (9.110 and the fact that  $g_0(\cdot, v) = 0$  quasi everywhere on  $\widetilde{K}_0$  for all  $v \in \mathbb{C} \setminus \widetilde{K}_0$ .

From (9.111) we deduce that

$$g_0(\cdot, \infty) = h_0 - \int_{\mathbb{C} \setminus \widetilde{K}_0} g_0(\cdot, v) d\widetilde{\sigma}_0(v). \tag{9.112}$$

Indeed, since  $\text{supp}(\widehat{\sigma}_0) \subset \widetilde{K}_0$ , the right-hand side of (9.112) is harmonic in  $\mathbb{C} \setminus \widetilde{K}_0$ , has an appropriate behavior at infinity, and is equal to zero quasi everywhere on  $\widetilde{K}_0$ . Hence, identity (9.112) holds true since the right-hand side of (9.112) satisfies the defining property (11.43) in Subsection 11.3 for the Green function  $g_0(\cdot, \infty) = g_{D_0}(\cdot, \infty)$ .

After this somewhat lengthy preparations we are ready to verify the individual statements of the lemma. We define

$$\sigma_0 := \widetilde{\sigma}_0|_{K_3 \setminus \widetilde{K}_0} = \widetilde{\sigma}_0|_{\mathbb{C} \setminus S_0} = -\sigma_1|_{\mathbb{C} \setminus S_0}. \tag{9.113}$$

The second equality in (9.113) is a consequence of (9.104), and the last one is identical with (9.108).

Representation (9.79) in the lemma follows directly from (9.112) and the introduction of the measure  $\sigma_0$  in (9.113). That the set  $\Sigma_0$  in (9.80) is a carrier of the measure  $\sigma_0$  is a consequence of (9.113) and (9.104).

Assertion (ii) is proved by (9.103) and (9.105).

Identity (9.85) follows directly from (9.76) and (9.77) in Definition 16 and the fact that for the Green functions we have  $g_j = 0$  quasi everywhere on  $K_j$ ,  $j = 1, 2$ , (cf. the defining property (11.43) in Subsection 11.3, further below).

Identity (9.86) is a consequence of (9.93) and the fact that  $h_1 = h_3 = \frac{1}{2} |g_1 - g_2|$  quasi everywhere on  $\mathbb{C} \setminus \text{Int}(K_3)$ .

Identity (9.87) follows from (9.113) and (9.104), and at last identity (9.88) is practically identical with (9.106).

Thus, only inequality (9.81) remains to be verified, and this will be done indirectly. Let us assume that  $\sigma_0 = 0$ . From (9.113), we then know that  $\sigma_1|_{\mathbb{C} \setminus S_0} = 0$ . It then is a consequence of (9.106) that the potential  $h_1 = p(\sigma_1; \cdot)$  is subharmonic in the domain  $(\mathbb{C} \setminus S_0) \cup \text{Int}(K_3)$ . From (9.77), (9.93), and (9.97) we know that

$$p(\sigma_1; z) = 0 \quad \text{for quasi every } z \in S_0 \setminus \text{Int}(K_3). \tag{9.114}$$

Since  $\sigma_1$  is of finite energy, it follows from (9.114) and the subharmonicity of  $h_1 = p(\sigma_1; \cdot)$  in  $(\mathbb{C} \setminus S_0) \cup \text{Int}(K_3)$  that  $h_1(z) \leq 0$  for all  $z \in \mathbb{C}$ . But the function  $h_1$  is non-negative by definition, and so we have shown that  $h_1 \equiv 0$ . But the last identity contradicts the assumption (9.52), and therefore assertion (9.81) is proved.  $\square$

With the proof of Lemma 14 all preparations are done for beginning the last step in the indirect proof of Proposition 7 which is the next lemma.

LEMMA 15. *Under the assumptions (9.48) and (9.50), we have*

$$\text{cap}(K_0) < c_0 \quad (9.115)$$

with  $K_0$  the compact set introduced in (9.58) of Definition 15, and  $c_0$  the constant introduced in (9.50).

PROOF. From (9.76) in Definition 16 and the assumption made in (9.50) we know that

$$h_0(z) = -\log(c_0) + O(z^{-1}) \quad \text{as } z \rightarrow \infty. \quad (9.116)$$

As in Lemma 14, we abbreviate the Green function  $g_{D_0}(\cdot, \cdot)$  by  $g_0(\cdot, \cdot)$ , and the special case  $g_0(\cdot, \infty)$  by  $g_0$ . From the representation of Green functions in Lemma 32 in Subsection 11.3, further below, we know that

$$g_0(z) = -\log(\text{cap}(K_0)) + O(z^{-1}) \quad \text{as } z \rightarrow \infty. \quad (9.117)$$

Hence, we have

$$\begin{aligned} \log \frac{\text{cap}(K_0)}{c_0} &= (h_0(z) - g_0(z))|_{z=\infty} \\ &= \int g_0(v, \infty) d\sigma_0(v), \end{aligned} \quad (9.118)$$

where the last equation follows from (9.79) in Lemma 14.

If we knew that  $\sigma_0$  were a purely negative measure, then we could get the desired estimate (9.115) very easily from (9.118). However, we cannot exclude that the measure  $\sigma_0$  contains a positive part. Therefore, we have to go a more complicated way for getting an estimation for the integral

$$I_0 := \int g_0(v, \infty) d\sigma_0(v). \quad (9.119)$$

The technical results in Lemma 14 will provides the basis for the analysis.

Using in (9.119) representation (9.79) from Lemma 14 leads us to the expression

$$I_0 = \int h_0 d\sigma_0 - \int \int g_0(v, w) d\sigma_0(v) d\sigma_0(w) =: I_1 - I_2. \quad (9.120)$$

From the positive definiteness of the Green function as a kernel in an energy formula, which has been stated in Lemma 35 in Subsection 11.3, further below, it follows together with (9.81) in Lemma 14 that

$$I_2 > 0. \quad (9.121)$$

In (9.120), there only remains the integral  $I_1 = \int h_0 d\sigma_0$  to be estimated. This will be done after some transformations. First, we make the following general remark: From Lemma 14 we know that the two measures  $\sigma_0$  and  $\sigma_1$  are both of finite energy. Because of Lemma 25 in Subsection 11.2, further below, we have  $\sigma_0(S) = \sigma_1(S) = 0$  for every measurable set  $S \subset \mathbb{C}$  of capacity zero. Consequently, integrals with respect to the measure  $\sigma_0$  or  $\sigma_1$  are equal if their integrands coincide quasi everywhere on a carrier of  $\sigma_0$  or  $\sigma_1$ , respectively.

As in (9.80) in Lemma 14, we denote by  $\Sigma_0$  the set  $(K_1 \cup K_2) \setminus \tilde{K}_0$ . Since  $\Sigma_0$  is a carrier of  $\sigma_0$ , we have

$$I_1 = \int h_0 d\sigma_0 = - \int_{\Sigma_0} h_0 d\sigma_1 \quad (9.122)$$

$$= - \int_{\Sigma_0} h_1 d\sigma_1 \quad (9.123)$$

$$= - \int h_1 d\sigma_1 + \int_{\tilde{K}_0 \cap \text{Int}(K_3)} h_1 d\sigma_1. \quad (9.124)$$

Indeed, the second equality in (9.122) follows from (9.80) and (9.87) in Lemma 14, the equality in (9.123) is a consequence of (9.85) in Lemma 14, and the equality in (9.124) follows from (9.86) in Lemma 14 and the fact that  $\Sigma_0 \cup (\tilde{K}_0 \cap \text{Int}(K_3))$  is a carrier of  $\sigma_1|_{\text{Int}(K_3)}$ .

From Lemma 29, part (ii), in Subsection 11.2, together with (9.87) and (9.81) from Lemma 14, we conclude that

$$\int h_1 d\sigma_1 = \int \int \log \frac{1}{|v-w|} d\sigma_1(v) d\sigma_1(w) > 0, \quad (9.125)$$

i.e., we have used the positive definiteness of the logarithmic kernel.

Since  $h_1 \geq 0$  by definition, it follows from (9.88) in Lemma 14 that

$$\int_{\tilde{K}_0 \cap \text{Int}(K_3)} h_1 d\sigma_1 \leq 0. \quad (9.126)$$

Putting (9.118), (9.120), (9.121), (9.124), (9.125), and (9.126) together, we conclude that

$$\log \frac{\text{cap}(K_0)}{c_0} < 0, \quad (9.127)$$

which proves (9.115).  $\square$

With the proof of Lemma 15, the preparations of the proof of Proposition 7 are completed. Despite of the complexity of some of the preparatory lemmas, the basic structure of the approach is straight forward. It starts with assumption (9.49), i.e., the assumption that there exist two essentially different admissible domains  $D_1$  and  $D_2$  with complements  $K_1$  and  $K_2$  of minimal capacity. Based on this assumption, a new admissible domain  $D_0$  with a complement  $K_0$  has been constructed in Definition 15, and it has then been shown in the last lemma that  $\text{cap}(K_0)$  is smaller than possible.

**Proof of Proposition 7.** The indirect proof of the proposition has been prepared by assumption (9.48). The introduction of the two sets  $K_0$  and  $D_0 = \mathbb{C} \setminus K_0$  in (9.58) and (9.59) of Definition 15 provide the basis for the falsification of assumption (9.48).

Indeed, in Lemma 13, it has been shown that for the domain  $D_0$  is admissible for Problem  $(f, \infty)$ , i.e.,  $D_0 \in \mathcal{D}(f, \infty)$ , and in Lemma 15, it then is proved that the newly constructed set  $K_0$  satisfies the inequality  $\text{cap}(K_0) < \text{cap}(K)$  for all  $K \in \mathcal{K}_0(f, \infty)$ , which contradicts the minimality (2.1) in Definition 2. Hence, assumption (9.48) is falsified, and Proposition 7 is proved.  $\blacksquare$

The construction of the two sets  $K_0$  and  $D_0$  in Definition 15 can be seen as a special case of a general type of set-theoretical convex combination, which will be elaborated further in Definition 17 in Subsection 9.5, below.

**9.4. The Unique Existence of an Extremal Domain.** In the present subsection we prove Theorem 2 and the two Propositions 1 and 2, which are all three concerned with the unique existence of an extremal domain for Problem  $(f, \infty)$ . With the two Propositions 6 and 7 in the last two Subsections 9.2 and 9.3, the main work for these proofs has already been done, we have only to put the different pieces together. We start with a technical lemma.

LEMMA 16. *The two sets*

$$K_0 := \bigcap_{K \in \mathcal{K}_0(f, \infty)} K \quad \text{and} \quad (9.128)$$

$$D_0 := \bigcup_{D \in \mathcal{D}_0(f, \infty)} D \quad (9.129)$$

are well defined, and we have

$$K_0 \in \mathcal{K}_0(f, \infty) \quad \text{and} \quad D_0 \in \mathcal{D}_0(f, \infty) \quad (9.130)$$

with the two sets  $\mathcal{K}_0(f, \infty)$  and  $\mathcal{D}_0(f, \infty)$  introduced in Definition 2.

PROOF. From Proposition 6 we know that  $\mathcal{K}_0(f, \infty) \neq \emptyset$ , hence, the sets  $K_0$  and  $D_0$  in (9.128) and (9.129) are well defined, and  $D_0$  is a domain with  $\infty \in D_0$ .

In order to prove (9.130), we have only to show that

$$D_0 \in \mathcal{D}(f, \infty), \quad (9.131)$$

since if we know that  $K_0 \in \mathcal{K}(f, \infty)$ , then the minimality condition (2.1) in Definition 2 follows immediately from (9.128) together with the monotonicity of the capacity (cf. Subsection 11.1). Relation (9.131) will be proved with the help of Proposition 5 of Subsection 9.1; for this purpose we have to show that the two assertions (i) and (ii) in Proposition 5 hold true for the domain  $D_0$  and the compact set  $K_0$ , respectively.

We start with assertion (i) in Proposition 5. For every  $z \in D_0$  there exists an admissible domain  $D_1 \in \mathcal{D}_0(f, \infty) \subset \mathcal{D}(f, \infty)$  with  $z \in D_1$ . Since assertion (i) holds true for  $D_1$ , it holds true also for the larger domain  $D_0$ .

Next, we prove that also assertion (ii) in Proposition 5 holds true for the set  $K_0$ . Let  $\gamma_0$  be an arbitrary Jordan curve of  $\Gamma_1 = \Gamma_1(f, \infty)$ . From assertion (ii) in Proposition 5 we know that

$$\gamma_0 \cap K \neq \emptyset \quad (9.132)$$

for all  $K \in \mathcal{K}_0(f, \infty)$ . In order to prove that (9.132) holds true also for the set  $K_0$ , we show in a first step that (9.132) holds true for the intersection  $K_{12} := K_1 \cap K_2$  of any two sets  $K_1, K_2 \in \mathcal{K}_0(f, \infty)$ . Indeed, let us assume that

$$\gamma_0 \cap K_{12} = \emptyset. \quad (9.133)$$

Let further  $R \subset \overline{\mathbb{C}} \setminus K_{12}$  be a ring domain as introduced in Lemma 2 with  $\gamma_0 \subset R$ . From Proposition 6 we know that

$$\text{cap}(K_1 \setminus K_2) = 0. \quad (9.134)$$

Hence, the set  $K_1 \setminus K_2$  cannot intersect the whole ring  $R$ , and consequently there exists a Jordan curve  $\gamma_1 \in \Gamma$  with

$$\gamma_1 \subset R \setminus (K_1 \setminus K_2) = R \setminus K_1 \quad (9.135)$$

that separates the two components  $A_1$  and  $A_2$  of  $\overline{\mathbb{C}} \setminus R$ . The equality in (9.135) is a consequence of  $K_1 = K_{12} \cup K_1 \setminus K_2$  and  $R \cap K_{12} = \emptyset$ .

From Lemma 2, part (ii), we know that  $\gamma_1 \sim \gamma_0$ . Hence, from the assumption  $\gamma_0 \in \Gamma_1$  we conclude also  $\gamma_1 \in \Gamma_1$ . On the other hand, it follows from (9.135) that  $\gamma_1 \cap K_1 = \emptyset$ , which then contradicts assertion (ii) in Proposition 5, and with this falsification of (9.133) we have proved that (9.132) holds true for  $K_{12}$ .

With the same argumentation as that applied to the intersection  $K_{12}$  of two elements from  $\mathcal{K}_0(f, \infty)$ , one can prove that relation (9.132) holds true also for an intersection of finitely many elements from  $\mathcal{K}_0(f, \infty)$ , i.e., we can prove that

$$\gamma_0 \cap (K_1 \cap \dots \cap K_m) \neq \emptyset \quad (9.136)$$

for an arbitrary  $m \in \mathbb{N}$  and arbitrarily chosen sets  $K_j \in \mathcal{K}_0(f, \infty)$ ,  $j = 1, \dots, m$ .

Let us now assume that relation (9.132) does not hold true for the set  $K_0$  of (9.128), i.e., we assume

$$\gamma_0 \cap \bigcap_{K \in \mathcal{K}_0(f, \infty)} K = \emptyset. \quad (9.137)$$

An infinite intersection of compact sets can be empty only if already a finite intersection is empty. However, such a possibility has been excluded in (9.136). Hence, we have proved that (9.132) holds true for the set  $K_0$ , and as a consequence, we have shown that assertion (ii) in Proposition 5 holds true for the set  $K_0$ .

Having verified the two assertions (i) and (ii) in Proposition 5 for  $D_0$  and  $K_0$ , it follows from the proposition that the domain  $D_0$  is admissible, i.e., (9.131) is proved, and the proof of the lemma is completed.  $\square$

We now come to the three proofs of the central results from Section 2.

**Proof of Theorem 2.** As minimal set  $K_0 = K_0(f, \infty)$  and as extremal domain  $D_0 = D_0(f, \infty)$  we choose the two sets introduced in (9.128) and (9.129). It follows immediately from Lemma 16 that  $K_0$  satisfies the three conditions (i), (ii), and (iii) in Definition 2. Hence, the existence side of Theorem 2 is established.

Uniqueness then follows immediately from (9.128) in Lemma 16.  $\blacksquare$

**Proof of Proposition 1.** The proof is a combination of Proposition 7 and Lemma 16. The first half-sentence in Proposition 1 has been proved in Proposition 7, and the second one is identical with (9.128) in Lemma 16.  $\blacksquare$

**Proof of Proposition 2.** If the function  $f$  has no branch points, then we have  $\Gamma_0 = \Gamma$  and  $\Gamma_1 = \emptyset$  in Definition 11. Hence, assertion (i) in Proposition 5 is trivially true, and therefore  $D_0 = D_0(f, \infty)$  is the largest domain to which the function  $f$  can be meromorphically extended. Such a domain can be denoted as the Weierstrass domain for meromorphic continuation of  $f$  if it is well-defined.

The domain  $D_0$  is identical with the Weierstrass domain  $W_f$  for analytic continuation of the function  $f$  plus all polar singularities of  $f$  that can be reached from within  $W_f$ , and which can be added without destroying the property that the resulting set is a domain. This completed the proof of Proposition 2.  $\blacksquare$

**9.5. A Convexity Property.** With the proof of Proposition 7 the main task of Subsection 9.3 had been done. However, in the present subsection, we will add an extension to Definition 15. It has already been mentioned after the proof Proposition 7 at the end of Subsection 9.3 that the construction of the set  $K_0$  in Definition 15 can be seen as a special case of a whole family of set-theoretic convex-combinations of the two sets  $K_1$  and  $K_2$  in Definition 15. The extended construction leads to a whole continuum of sets  $K_h$  with  $h \in [0, 1]$ , and for the capacity of these sets  $K_h$  we get an interesting inequality that generalizes inequality (9.115) in Lemma 15. These extended results are certainly of independent interest, but they are also needed in Subsection 10.1, below. The main properties of the new sets  $K_h$ ,  $h \in [0, 1]$ , are proved in Theorem 13.

DEFINITION 17. *For two arbitrarily chosen admissible domains  $D_0, D_1 \in \mathcal{D}(f, \infty)$  with corresponding compact sets  $K_j = \overline{\mathbb{C}} \setminus D_j \in \mathcal{K}(f, \infty)$ ,  $j = 0, 1$ , that satisfy*

$$\text{cap}(K_j) > 0, \quad j = 0, 1, \quad (9.138)$$

*we define a family of domains  $D_h \subset \overline{\mathbb{C}}$ ,  $0 \leq h \leq 1$ , (which will turn out to be admissible domains) together with a family of corresponding compact sets  $K_h = \overline{\mathbb{C}} \setminus D_h$ ,  $0 \leq h \leq 1$ , in a way that generalizes Definition 15. For  $0 \leq h \leq 1$  we define:*

$$S_h := \overline{\{z \in \overline{\mathbb{C}} \mid (1-h)g_0(z) = h g_1(z)\}}, \quad (9.139)$$

$$K_3 := \widehat{K_0 \cup K_1}, \quad (9.140)$$

$$\tilde{K}_h := \widehat{S_h \cap K_3}, \quad (9.141)$$

$$K_{0,h} := \{z \in K_0 \mid (1-h)g_0(z) > h g_1(z)\}, \quad (9.142)$$

$$K_{1,h} := \{z \in K_1 \mid (1-h)g_0(z) < h g_1(z)\}, \quad (9.143)$$

$$K_h := \tilde{K}_h \cup K_{0,h} \cup K_{1,h}, \quad (9.144)$$

$$D_h := \overline{\mathbb{C}} \setminus K_h \quad (9.145)$$

*with  $g_j = g_{D_j}(\cdot, \infty)$  the Green function in the domain  $D_j$ ,  $j = 0, 1$ .*

It is immediate that Definition 17 is a generalization of Definition 15. The role of the two input sets  $K_1$  and  $K_2$  in Definition 15 is now played by the two sets  $K_0$  and  $K_1$ , respectively. The two sets  $K_0$  and  $D_0$  in (9.58) and (9.59) of Definition 15 now correspond to the two sets  $K_{1/2}$  and  $D_{1/2}$ , respectively, in the new terminology of Definition 17.

Another generalization in Definition 17 concerns the assumptions made with respect to the two compact input sets  $K_0$  and  $K_1$ . While in Definition 15 the input sets have been assumed to be of minimal capacity, this assumption has been dropped without replacement in the extended definition.

The change of notation with respect to the input sets  $K_1$  and  $K_2$  in Definition 15 into the sets  $K_1$  and  $K_2$  in Definition 17 was necessary, and has the advantage that in the family of the newly defined sets  $K_h$ ,  $h \in [0, 1]$ , the two special sets  $K_0$  and  $K_1$  coincide with the two input sets  $K_0$  and  $K_1$  in Definition 17, which can easily be verified.

In the next theorem we prove that the newly defined domains  $D_h$ ,  $h \in [0, 1]$ , are all admissible for Problem  $(f, \infty)$ , and most importantly, we prove that the functional  $\log \text{cap}(K_h)$  depends on the index  $h \in [0, 1]$  in a strictly convex manner.

THEOREM 13. (i) Under the assumptions of Definition 17 we have

$$D_h \in \mathcal{D}(f, \infty) \quad \text{and} \quad K_h \in \mathcal{K}(f, \infty) \quad \text{for all} \quad h \in [0, 1] \quad (9.146)$$

with  $\mathcal{K}(f, \infty)$  and  $\mathcal{D}(f, \infty)$  the sets introduced in Definition 1.

(ii) If in addition to the assumptions of Definition 17 we assume that

$$\text{cap}((K_1 \setminus K_0) \cup (K_0 \setminus K_1)) > 0, \quad (9.147)$$

then we have

$$\log \text{cap}(K_h) < (1 - h) \log \text{cap}(K_0) + h \log \text{cap}(K_1) \quad \text{for} \quad 0 < h < 1. \quad (9.148)$$

(iii) Under the assumptions of Definition 17 we have the following continuity property: For any  $h_0 \in [0, 1]$  and for any open set  $U \subset \overline{\mathbb{C}}$  with  $K_{h_0} \subset U$ , there exists a neighborhood  $V_0 \subset \mathbb{R}$  of  $h_0$  such that

$$K_h \subset U \quad \text{for all} \quad h \in V_0 \cap [0, 1]. \quad (9.149)$$

REMARK 3. Assertion (iii) in Theorem 13 means that in the Hausdorff metric the compact sets  $K_h$  depend continuously on the parameter  $h \in [0, 1]$ .

PROOF. Definition 15 has been the backbone of the proof of the essential uniqueness of minimal sets in Proposition 7 in Subsection 9.3. The important Lemma 15 in Subsection 9.3 can be seen as a special case of the convexity relation (9.148). It turns out that the proof of Theorem 13 is based on almost the same argumentations and techniques as those applied in the proof of Proposition 7, therefore we will now very closely follow the different stages of argumentations used in Subsection 9.3. As a consequence, we can shorten the proof of Theorem 13 considerably.

As a general policy, we will reformulate the content of lemmas and definitions from Subsection 9.3 in such a way that it satisfies the needs of our new situation, but we will use shortcuts and will not repeat all details. Often it is only necessary to replace the difference  $g_1 - g_2$  of the two Green functions  $g_1$  and  $g_2$  from Definition 15 by the convex combination  $(1 - h)g_0 + hg_1$  of the two Green functions  $g_0$  and  $g_1$  from Definition 17. This change is evidently suggested by (9.139) in Definition 17.

Thus, for instance, the difference  $d := g_1 - g_2$  in (9.60) will now be replaced by

$$d(z) := ((1 - h)g_0 + hg_1)(z) \quad \text{for} \quad h \in [0, 1] \quad \text{and} \quad z \in \overline{\mathbb{C}}. \quad (9.150)$$

By using the same replacement repeatedly, one can transform all elements of Definition 15 into those of Definition 17. In the same way the auxiliary definitions in the two Lemmas 11 and 12 can be adapted to the new situation, and like in Lemma 13, one can prove that

$$K_h \in \mathcal{K}(f, \infty) \quad \text{and} \quad D_h \in \mathcal{D}(f, \infty) \quad \text{for all} \quad h \in [0, 1] \quad (9.151)$$

with  $K_h$  and  $D_h$ ,  $h \in [0, 1]$ , the sets introduced in (9.144) and (9.145), respectively. The last conclusion proves assertion (i) of Theorem 13.

Analogously to (9.76) and (9.77) in Definition 16, we now introduce the two auxiliary functions  $h_0$  and  $h_1$  by defining

$$h_0(z) := \begin{cases} ((1 - h)g_0 + hg_1)(z) & \text{for} \quad z \in \overline{\mathbb{C}} \setminus K_3, \\ |(1 - h)g_0 - hg_1|(z) & \text{for} \quad z \in K_3 \setminus \text{Int}(\tilde{K}_h), \\ 0 & \text{for} \quad z \in \text{Int}(\tilde{K}_h), \end{cases} \quad (9.152)$$

$$h_1(z) := \begin{cases} |(1-h)g_0 - h g_1|(z) & \text{for } z \in \overline{\mathbb{C}} \setminus K_3, \\ ((1-h)g_0 + h g_1)(z) & \text{for } z \in K_3 \setminus \text{Int}(\tilde{K}_h), \\ ((1-h)\widehat{g_0} + h g_1)(z) & \text{for } z \in \text{Int}(\tilde{K}_h) \end{cases} \quad (9.153)$$

for  $h \in [0, 1]$  with sets  $K_3$  and  $\tilde{K}_h$  that have been defined in (9.140) and (9.141). In (9.153), the expression  $(1-h)\widehat{g_0} + h g_1$  denotes the solution of the Dirichlet problem in each component  $C$  of the interior  $\text{Int}(\tilde{K}_h)$  of  $\tilde{K}_h$  with  $((1-h)g_0 + h g_1)|_{\partial \tilde{K}_0}$  as boundary function. Both functions  $h_0$  and  $h_1$  depend on the parameter  $h$ .

Representations for the functions  $h_0$  and  $h_1$  can be proved in the same way as has been done in Lemma 14. Like in (9.79), we have a representation for  $h_0$  that now takes the form

$$h_0 = g_h(\cdot, \infty) + \int g_h(\cdot, v) d\sigma_0(v) \quad (9.154)$$

with  $g_h(\cdot, \cdot)$  the Green function of the domain  $D_h$ ,  $h \in [0, 1]$ . For the measure  $\sigma_0$  in (9.154) we have

$$\text{supp}(\sigma_0) \subset (K_0 \cup K_1) \setminus \text{Int}(\tilde{K}_h), \quad (9.155)$$

$$\sigma_0(\overline{\mathbb{C}} \setminus \Sigma_0) = 0 \quad \text{for } \Sigma_0 := (K_1 \cup K_2) \setminus \tilde{K}_0, \quad (9.156)$$

and if  $0 < h < 1$ , then we have

$$\sigma_0 \neq 0 \quad (9.157)$$

as a consequence of assumption (9.147). These conclusions can be proved in exactly the same way as the corresponding assertions have been proved in the proof of Lemma 14.

The assertions (ii) and (iii) of Lemma 14 hold true also in the new situation if one substitutes the sets  $S_0$ ,  $K_1$ ,  $K_2$ ,  $\tilde{K}_0$  by the sets  $S_h$ ,  $K_0$ ,  $K_1$ ,  $\tilde{K}_h$ . The sets  $S_h$  and  $\tilde{K}_h$  have been introduced in Definition 17. The proof of Lemma 14 is quite long and involved, and the same is true in the new situation if all details are taken into consideration. Since everything can be done in practically the same way as before, we will skip all details here.

In the new situation of Definition 17 the convexity relation (9.148) is the analog of the inequality (9.115) in Lemma 15, and its proof can be done in quite the same way as that of Lemma 14. Like in (9.117) and (9.118), from (9.154) and (9.152) together with representation (11.45) for Green functions in Lemma 32 of Subsection 11.3, further below, it follows that we have

$$\begin{aligned} \log \text{cap}(K_h) - [(1-h) \log \text{cap}(K_0) + h \log \text{cap}(K_1)] &= \\ &= (h_0(z) - g_h(z, \infty))|_{z=\infty} \\ &= \int g_h(v, \infty) d\sigma_0(v). \end{aligned} \quad (9.158)$$

In the verification of (9.158), representation (11.45) has been applied to the Green functions  $g_0$ ,  $g_1$ , and  $g_h$ . By  $g_h = g_h(\cdot, \cdot)$  we denote the Green function of the domain  $D_h$  for  $h \in [0, 1]$ .

In the same way as has been done in the proof of Lemma 14, it is then shown that

$$\int g_h(v, \infty) d\sigma_0(v) < 0 \quad (9.159)$$



if, and only if,  $0 < h < 1$ . Together with (9.158), the last conclusion proves assertion (ii) of Theorem 13.

It remains to prove assertion (iii), which will be done indirectly. We assume that there exist  $h_0 \in [0, 1]$  and an open set  $U \subset \mathbb{C}$  with  $K_{h_0} \subset U$  such that there exist  $h_n \in [0, 1]$ ,  $n \in \mathbb{N}$ , with

$$h_n \rightarrow h_0 \quad \text{as } n \rightarrow \infty \quad \text{and} \quad K_{h_n} \setminus U \neq \emptyset \quad \text{for all } n \in \mathbb{N}. \quad (9.160)$$

Without loss of generality, we can then select  $x_n \in K_{h_n} \setminus U$ ,  $n \in \mathbb{N}$ , such that

$$x_n \rightarrow x_0 \quad \text{as } n \rightarrow \infty. \quad (9.161)$$

From (9.141) - (9.144) it follows that

$$x_0 \in K_3 \setminus U. \quad (9.162)$$

We shall prove that assertion (9.162) is contradictory, which then implies that assertion (iii) of the theorem holds true. For disproving assertion (9.162) we distinguish three different cases.

**Case 1:** We assume that  $x_0 \notin K_0 \cup K_1$ . Then both Green functions  $g_0$  and  $g_1$  are harmonic and continuous in a neighborhood of  $x_0$ . As a consequence, it follows from  $x_n \in K_{h_n}$  for  $n \in \mathbb{N}$  that also  $x_0 \in K_{h_0}$ , which then disproves assertion (9.162) since  $K_{h_0} \subset U$ .

We will give some more details of these last conclusions. From (9.140) - (9.144) together with  $x_n \in K_{h_n}$  and  $x_0 \notin K_0 \cup K_1$ , we deduce that  $x_n \in \tilde{K}_{h_n}$  for  $n \in \mathbb{N}$  sufficiently large. From (9.139) and (9.141) we then know that

$$(1 - h_n) g_0(x_n) = h_n g_1(x_n) \quad \text{for } n \in \mathbb{N} \text{ sufficiently large.} \quad (9.163)$$

From the continuity of  $g_0$  and  $g_1$  together with the limits in (9.160) and (9.161) we deduce from (9.163) that

$$(1 - h_0) g_0(x_0) = h_0 g_1(x_0), \quad (9.164)$$

which implies that  $x_0 \in K_{h_0}$ , as stated above.

**Case 2:** Let us now assume that  $x_0 \in K_0 \cap K_1$ . From (9.139) - (9.144) it follows that  $K_0 \cap K_1 \subset K_h$  for all  $h \in [0, 1]$ . Consequently, we have  $x_0 \in K_{h_0}$ , and because of  $K_{h_0} \subset U$  this disproves (9.162).

**Case 3:** We assume that  $x_0 \in (K_0 \cup K_1) \setminus (K_0 \cap K_1)$ . Because of symmetry, we can assume without loss of generality that

$$x_0 \in K_0 \setminus K_1. \quad (9.165)$$

From (9.165) it follows that the Green function  $g_1$  is positive and harmonic in a neighborhood of  $x_0$ . Further, we can assume without loss of generality that

$$x_n \notin K_1 \quad \text{for all } n \in \mathbb{N}. \quad (9.166)$$

Because of (9.139) - (9.144), we can conclude from (9.166) and  $x_n \in K_{h_n}$  that

$$(1 - h_n) g_0(x_n) \geq h_n g_1(x_n), \quad (9.167)$$

which implies that

$$\liminf_{n \rightarrow \infty} d_0(x_n) \geq 0 \quad (9.168)$$

for  $d_0 := (1 - h_0) g_0 - h_0 g_1$ . In the last conclusion we have used the fact that the difference  $g_0 - g_1$  is bounded in a neighborhood of  $x_0$ .

The function  $d_0$  is subharmonic and non-constant in the neighborhood of  $x_0$ .

If  $d_0(x_0) > 0$ , then it follows from (9.165) and (9.142) that  $x_0 \in K_{0,h_0} \subset K_{h_0}$ . If, on the other hand,  $d_0(x_0) = 0$ , then it follows from (9.139) and (9.141) that  $x_0 \in \tilde{K}_{h_0} \subset K_{h_0}$ . If  $d_0(x_0) < 0$ , then it follows from (9.168) that in each neighborhood of  $x_0$  there exists a point  $\tilde{x}_0$  with  $d_0(\tilde{x}_0) = 0$ , which, because of (9.139) and (9.141), again implies that  $x_0 \in \tilde{K}_{h_0} \subset K_{h_0}$ . Hence, we have proved that  $x_0 \in K_{h_0}$ , which disproves assertion (9.162) because of  $K_{h_0} \subset U$ .

Since (9.162) has been disproved for all three cases, the proof of assertion (iii) of Theorem 13 is completed.  $\square$

## 10. Proofs II

In the present section we prove all results that have been stated in the Sections 4, 5, and 7. In the first subsection we deal with the special case of an algebraic function  $f$ . The results proved there are of independent interest, and this is especially true for Theorem 15 towards the end of the subsection. But besides of that they are also an essential preparation for the proofs of the main results from Sections 4 and 5, which will be given in Subsection 10.2. Results from Section 7 are proved in the last two subsections.

**10.1. Algebraic Functions.** The particularity of algebraic functions  $f$  with respect to our investigation is the fact that they possess only finitely many branch points and no other types of non-polar singularities. As a consequence, the structure of the minimal set  $K_0(f, \infty)$  is in many respects special and also much simpler to describe than this is the case in general. All functions  $f$  in the Examples 6.1 - 6.5 of Section 6 are algebraic, and these examples are illustrations of what we can expect on special results. Since there exist only finitely many branch points, we have a direct connection between Problem  $(f, \infty)$  and a certain type of Problem 3, which has already been discussed in Subsection 8.1. Details of the connection will be a major topic in the present subsection; another one will be the role of rational quadratic differentials, which are in some sense typical for Problem  $(f, \infty)$  with  $f$  being an algebraic function.

**10.1.1. Sets of Minimal Hyperbolic Capacity.** In the present subsection we investigate a special case of Problem 3 from Subsection 8.1.

**DEFINITION 18.** Let  $A = \{a_1, \dots, a_n\} \subset \mathbb{D}$  be a set of  $n \geq 2$  distinct points. The task to find a continuum  $K \subset \mathbb{D}$  with the property that

$$a_j \in K \quad \text{for } j = 1, \dots, n, \quad (10.1)$$

and that the condenser capacity  $\text{cap}(K, \partial\mathbb{D})$  is minimal among all continua  $K \subset \mathbb{D}$  that satisfy (10.1) is called Problem  $(A, \mathbb{D})$ . Its solution is denoted by  $K_0 = K_0(A, \mathbb{D})$ .

For a definition of the condenser capacity  $\text{cap}(K, V)$  with arbitrary compact sets  $K, V \subset \mathbb{C}$  we refer to Chapter II.5 in [27] or to [1]. In the special with  $K \subset \mathbb{D}$  and  $V := \partial\mathbb{D}$ ,  $\text{cap}(K, \partial\mathbb{D})$  is also known as the hyperbolic capacity of  $K$  in  $\mathbb{D}$ . For more details see [42]. Because of this terminology, the solution  $K_0(A, \mathbb{D})$  of Problem  $(A, \mathbb{D})$  is also called set of minimal hyperbolic capacity, and Problem  $(A, \mathbb{D})$  can be seen as the hyperbolic analogue of Chebotarev's Problem, which has been discussed in Section 8 as Problem 1.

Problem  $(A, \mathbb{D})$  can also be seen as a special case of Problem 3 from Section 8.1. This connection is established in the next proposition.

**PROPOSITION 8.** *A continuum  $K_0 \subset \mathbb{D}$  is a solution of Problem  $(A, \mathbb{D})$  if, and only if, the pair of sets  $(K_0, K_0^{-1})$  is a solution of Problem 3 formulated with the two sets of points  $A = \{a_1, \dots, a_n\} \subset \mathbb{D}$  and  $B = \{b_1, \dots, b_n\} := A^{-1} \subset \overline{\mathbb{C}} \setminus \mathbb{D}$ , i.e.,  $b_j := 1/\overline{a_j}$  for  $j = 1, \dots, n$ . By  $S^{-1}$  we denote the reflection of a set  $S$  on the unit circle  $\partial\mathbb{D}$ .*

Proposition 8 follows from Theorem 3.1 in [13], and the relevant elements in its deduction are also assembled in Theorem 14, below.

The capacity  $\text{cap}(K, \partial\mathbb{D})$  depends only on the outer boundary of  $K \subset \mathbb{D}$ , and therefore we have

$$\text{cap}(K, \partial\mathbb{D}) = \text{cap}(\widehat{K}, \partial\mathbb{D}), \quad (10.2)$$

where  $\widehat{K}$  denotes the polynomial-convex hull of  $K$ . If  $K \subset \mathbb{D}$  is a continuum with  $K = \widehat{K}$ , then  $\mathbb{D} \setminus K$  is a ring domain, and in this special case  $\text{cap}(K, \partial\mathbb{D})$  is the reciprocal of the modulus of this ring domain (cf. [1]). If  $K$  is not reduced to a single point, then there exists  $1 < r < \infty$  and a bijective conformal map

$$\varphi : \mathbb{D} \setminus K \longrightarrow \{1 < |z| < r\} \quad (10.3)$$

with  $\varphi(1) = 1$ . The modulus of  $\mathbb{D} \setminus K$  is then defined as  $\log(r)$ , and consequently, we have  $\text{cap}(K, \partial\mathbb{D}) = 1/\log(r)$ .

The function

$$p(z) := \begin{cases} 0 & \text{for } z \in K \\ \log |\varphi(z)| & \text{for } z \in \mathbb{D} \setminus K \\ \log(r) = 1/\text{cap}(K, \partial\mathbb{D}) & \text{for } z \in \overline{\mathbb{C}} \setminus \mathbb{D}. \end{cases} \quad (10.4)$$

is known as the equilibrium potential of the condenser  $(K, \partial\mathbb{D})$ . It is harmonic in  $\mathbb{D} \setminus K$ , and continuous throughout  $\overline{\mathbb{C}}$ .

Problem  $(A, \mathbb{D})$  has a unique solution  $K_0 = K_0(A, \mathbb{D}) \subset \mathbb{D}$ . The continuum  $K_0$  can be described very nicely by critical trajectories of a quadratic differential. In the next theorem we assemble these results together with other properties of the solution  $K_0(A, \mathbb{D})$ , which will be important for our further investigations. All results of the theorem have been proved in Chapter 3 of [13].

**THEOREM 14.** ([13], Theorem 3.1) *Let  $A = \{a_1, \dots, a_n\} \subset \mathbb{D}$  be a set of  $n \geq 2$  distinct points.*

(i) *There exists a continuum  $K_0 = K_0(A, \mathbb{D}) \subset \mathbb{D}$ , which is the unique solution of Problem  $(A, \mathbb{D})$  as introduced in Definition 18.*

(ii) *There exist  $n - 2$  points  $b_1, \dots, b_{n-2} \in \mathbb{D}$  such that the continuum  $K_0$  is the union of the closed critical trajectories of the quadratic differential*

$$q(z) dz^2 \quad \text{with} \quad q(z) := \frac{(z - b_1) \dots (z - b_{n-2})(1 - \overline{b_1}z) \dots (1 - \overline{b_{n-2}}z)}{(z - a_1) \dots (z - a_n)(1 - \overline{a_1}z) \dots (1 - \overline{a_n}z)} \quad (10.5)$$

*in  $\mathbb{D}$ . There exist only finitely many critical trajectories.*

(iii) *The equilibrium potential  $p_0$  from (10.4) which is associated with the extremal condenser  $(K_0, \partial\mathbb{D})$  satisfies relation*

$$\left( \frac{\partial}{\partial z} p_0(z) \right)^2 = \frac{1}{4} q(z) \quad \text{for } z \in \overline{\mathbb{D}} \quad (10.6)$$

with  $\partial/\partial z$  denoting complex differentiation. The potential  $p_0$  can be extended to a harmonic function in  $\overline{\mathbb{C}} \setminus (K_0 \cup K_0^{-1})$  by a reflection on the unit circle  $\partial\mathbb{D}$ , and relation (10.6) holds true throughout  $\overline{\mathbb{C}}$  for the harmonic extension of  $p_0$ .

(iv) We have

$$p_0(z) := \begin{cases} 1/\text{cap}(K_0, \partial\mathbb{D}) & \text{for } z \in \overline{\mathbb{C}} \setminus \mathbb{D} \\ 0 & \text{for } z \in K_0 \end{cases} \quad (10.7)$$

and

$$\frac{1}{2\pi} \oint_{\partial\mathbb{D}} \frac{\partial}{\partial n} p_0(\zeta) ds_\zeta = -1 \quad (10.8)$$

with  $\partial/\partial n$  the inward showing normal derivative on  $\partial\mathbb{D}$  and  $ds$  the line element on  $\partial\mathbb{D}$ .

**10.1.2. Problem  $(f, \infty)$  for Algebraic Functions.** In the present subsection we study Problem  $(f, \infty)$  for an algebraic function  $f$ . We will shed light on the connection between this problem and certain aspects of Problem  $(A, \mathbb{D})$  from Definition 18.

Let  $f$  be an algebraic function and assume that this function is meromorphic at infinity. Algebraic functions have only finitely many singularities, and the only non-polar singularities are branch points. Hence, in Problem  $(f, \infty)$ , only a finite number of points is of critical relevance. By  $E_0 \subset K_0(f, \infty)$  we denote the (finite) set of branch points of the function  $f$  that can be reached on the minimal set  $K_0(f, \infty)$  by meromorphic continuation of  $f$  from within the extremal domain  $D_0(f, \infty)$ . In the discussion of the examples in Section 6, this type of branch points have been called the active branch points for the determination of the minimal set  $K_0(f, \infty)$ . The sets  $D_0(f, \infty)$  and  $K_0(f, \infty)$  have been introduced in Definition 2.

**LEMMA 17.** *Let  $f$  be an algebraic function that is meromorphic at infinity. Then the minimal set  $K_0(f, \infty)$  for Problem  $(f, \infty)$  has only finitely many components, which we denoted by  $K_1, \dots, K_m$ , i.e., we have*

$$K_0(f, \infty) = K_1 \cup \dots \cup K_m. \quad (10.9)$$

*Each component  $K_j$ ,  $j = 1, \dots, m$ , contains at least two branch points of  $f$ . The Green function  $g_{D_0}(\cdot, \infty)$  in the extremal domain  $D_0 = D_0(f, \infty)$  has only finitely many critical points, and we have*

$$g_{D_0}(z, \infty) = 0 \quad \text{for all } z \in K_0(f, \infty). \quad (10.10)$$

**PROOF.** It follows from the two conditions (ii) and (iii) in Definition 2 that each component of  $K_0(f, \infty)$  has to contain at least one non-polar singularity of  $f$ , and the single-valuedness of  $f$  in  $D_0(f, \infty)$  then further implies that each component of  $K_0(f, \infty)$  has to contain at least two branch points.

After this conclusion, all other assertions of the lemma follow directly from of the finiteness of the set  $E_0$  and the fact that a continuum has no irregular points with respect to the Dirichlet problem (cf. Subsection 11.3, further below).  $\square$

Based on Lemma 17, we divide the set  $E_0$  into  $m$  subsets  $E_j := E_0 \cap K_j$ ,  $j = 1, \dots, m$ . I.e., we have

$$E_0 = E_1 \cup \dots \cup E_m \text{ with } E_j \subset K_j \text{ for } j = 1, \dots, m. \quad (10.11)$$

For  $c > 0$  we define the open set

$$U_c := \{z \in \mathbb{C} \mid g_{D_0}(z, \infty)(z) < c\} \quad (10.12)$$

with the Green function  $g_{D_0}(\cdot, \infty)$  in  $D_0 = D_0(f, \infty)$ . Since  $g_{D_0}(\cdot, \infty)$  has only finitely many critical points in  $D_0$ , the open set  $U_c$  consists of exactly  $m$  components for  $c > 0$  sufficiently small. The number  $m$  is the same as that in (10.9).

LEMMA 18. *Let the same assumptions hold true as in Lemma 17. Then a constant  $c_0 > 0$  can be chosen in such a way that the open set  $U_0 := U_{c_0}$  from (10.12) has the following properties:*

- (i)  $\overline{U_0}$  contains no critical point of the Green function  $g_{D_0}(\cdot, \infty)$ .
- (ii)  $U_0$  consists of exactly  $m$  components  $U_j$ ,  $j = 1, \dots, m$ , i.e.,

$$U_0 = U_1 \cup \dots \cup U_m, \quad (10.13)$$

and we have

$$K_j \subset U_j, \quad \text{for } j = 1, \dots, m \quad (10.14)$$

with  $K_j$  introduced in (10.9).

- (iii) Each component  $U_j$ ,  $j = 1, \dots, m$ , in (10.13) is simply connected, and  $\partial U_j$  is an analytic Jordan curve.

PROOF. All three assertions of the lemma are rather immediate. Because of (10.10),  $U_0$  is an open neighborhood of  $K_0(f, \infty)$ , and we can shrink  $U_0$  as close to  $K_0(f, \infty)$  as we wish. The first assertions (i) follows from the fact that the Green function  $g_{D_0}(\cdot, \infty)$  has only finitely many critical points in  $D_0(f, \infty)$ .

The two other assertions (ii) and (iii) follow then immediately from (10.10) for  $c_0 > 0$  sufficiently small.  $\square$

In the next proposition we establish the connections between Problem  $(f, \infty)$  and problems of the type of Problem  $(A, \mathbb{D})$ . These connections are the main topic in the present subsection.

PROPOSITION 9. *Let  $f$  be an algebraic function that is meromorphic at infinity, and let  $K_0$  be the minimal set  $K_0(f, \infty)$  of Definition 2 for Problem  $(f, \infty)$ . Let further the sets  $K_j$ ,  $E_j$ , and  $U_j$ ,  $j = 1, \dots, m$ , be defined as in (10.9), (10.11), and (10.13), respectively, and let  $\varphi_j : U_j \rightarrow \mathbb{D}$  be Riemann mapping functions,  $j = 1, \dots, m$ . We set*

$$A_j := \varphi_j(E_j), \quad K_{0,j} := \varphi_j(K_j), \quad \alpha_j := \omega_{K_0}(K_j), \quad j = 1, \dots, m, \quad (10.15)$$

with  $\omega_{K_0}$  denoting the equilibrium distribution on  $K_0$  as introduced in Subsection 11.2, further below. The following two assertions hold true:

- (i) For each  $j = 1, \dots, m$ , the set  $K_{0,j}$  is the minimal set  $K_0(A_j, \mathbb{D})$  that solves Problem  $(A_j, \mathbb{D})$  from Definition 18, which has been analyzed in Theorem 14.
- (ii) For each  $j = 1, \dots, m$ , we have

$$g_{D_0}(z, \infty)(z) = \alpha_j(p_{0,j} \circ \varphi_j)(z) \quad \text{for } z \in U_j, \quad (10.16)$$

where  $p_{0,j}$  is the equilibrium potential (10.4) for the extremal condenser  $(K_{0,j}, \mathbb{D})$  of Problem  $(A_j, \mathbb{D})$ , and  $g_{D_0}(\cdot, \infty)$  is the Green function in the extremal domain  $D_0(f, \infty)$ .

The practical significance of Proposition 8 is that it shows a possibility to transplant specific properties of the solutions  $K_0(A_j, \mathbb{D})$  of the Problems  $(A_j, \mathbb{D})$ ,  $j = 1, \dots, m$ , to the solution of Problem  $(f, \infty)$ .

PROOF. We start with assertion (i), which will be proved indirectly. For this purpose, we assume that at least one of the sets  $K_{0,j}$ ,  $j = 1, \dots, m$ , is not a minimal solution  $K_0(A_j, \mathbb{D})$  of Problem  $(A_j, \mathbb{D})$ . Without loss of generality we can assume that

$$K_{0,1} \neq \tilde{K}_{0,1} := K_0(A_1, \mathbb{D}). \quad (10.17)$$

We define

$$\tilde{K}_1 := \varphi_1^{-1}(\tilde{K}_{0,1}), \quad \tilde{K}_0 := (K_0 \setminus K_1) \cup \tilde{K}_1, \quad \tilde{D}_0 := \overline{\mathbb{C}} \setminus \tilde{K}_0, \quad (10.18)$$

and show that the domain  $\tilde{D}_0$  is admissible for Problem  $(f, \infty)$ , i.e.,

$$\tilde{D}_0 \in \mathcal{D}(f, \infty). \quad (10.19)$$

From (10.17) we then deduce that

$$\text{cap}(\tilde{K}_0) < \text{cap}(K_0(f, \infty)). \quad (10.20)$$

If (10.17) and (10.19) are proved, then with (10.20) we have a contradiction since, because of (10.19), inequality (10.20) clearly contradicts the minimality (2.1) in Definition 2 of the set  $K_0 = K_0(f, \infty)$ . The contradiction shows that assumption (10.17) is false, and therefore assertion (i) is proved.

We start with the proof of (10.19). Using Cauchy's formula, one can rewrite the function  $f$  as

$$f = f_1 + \dots + f_m \quad (10.21)$$

with each  $f_j$ ,  $j = 1, \dots, m$ , being meromorphic and single-valued in the simply connected domain  $\overline{\mathbb{C}} \setminus K_j$ . Since the sets  $U_1, \dots, U_m$  are disjoint, we have only to consider the function  $f_1$  if we want to understand the changes in the global behavior of  $f$  that are caused by the exchange of the sets  $K_1$  and  $\tilde{K}_1$  that is defined by (10.18).

From (10.11), (10.15), (10.17), and (10.18), we know that both sets  $K_1$  and  $\tilde{K}_1$  contain the same set  $E_1$  of branch points of the function  $f$  on  $K_1$ . Consequently, the function  $f_1$  can be continued meromorphically throughout the whole domain  $\overline{\mathbb{C}} \setminus \tilde{K}_1$ . Since the domain  $\overline{\mathbb{C}} \setminus \tilde{K}_1$  is simply connected, it follows from the Monodromy Theorem that the continuation of  $f_1$  is single-valued in  $\overline{\mathbb{C}} \setminus \tilde{K}_1$ , and consequently, the function  $f$  has also a single-valued meromorphic continuation to the domain  $\tilde{D}_0$ , which proves (10.19).

In order to prove (10.20), we observe first that from the uniqueness of the solution  $\tilde{K}_{0,1} := K_0(A_1, \mathbb{D})$  of Problem  $(A_1, \mathbb{D})$ , which has been established in Theorem 14, it follows that

$$\text{cap}(\tilde{K}_{0,1}, \partial\mathbb{D}) < \text{cap}(K_{0,1}, \partial\mathbb{D}) \quad (10.22)$$

with  $K_{0,1}$  defined in (10.15). Notice that  $A_1 \subset K_{0,1}$ . We shall now show that inequality (10.22) implies (10.20).

Indeed, let  $p_{01}$  and  $\tilde{p}_{01}$  be the equilibrium potentials (10.4) of the two condensers  $(K_{0,1}, \mathbb{D})$  and  $(\tilde{K}_{0,1}, \mathbb{D})$ , respectively. From (10.4) it follows that

$$p_{01}(z) = \frac{1}{\text{cap}(K_{0,1}, \partial\mathbb{D})}, \quad \tilde{p}_{01}(z) = \frac{1}{\text{cap}(\tilde{K}_{0,1}, \partial\mathbb{D})} \quad \text{for } z \in \partial\mathbb{D}. \quad (10.23)$$

From the definition of the open set  $U_0$  in Lemma 18 together with the properties of the Green function  $g_{D_0}(\cdot, \infty)$ , the definition of the mapping  $\varphi_1 : U_1 \rightarrow \mathbb{D}$ , the set  $K_{0,1}$ , and the number  $\alpha_1$ , which has been introduced in (10.15), we then deduce that the function

$$\check{p}_{01} := \frac{1}{\alpha_1} g_{D_0}(\cdot, \infty) \circ \varphi_1^{-1} \quad (10.24)$$

has the following four properties: (i) The function  $\check{p}_{01}$  is harmonic in  $\mathbb{D} \setminus K_{0,1}$ . (ii) We have  $\check{p}_{01}(z) = 0$  for all  $z \in K_{0,1}$ . (iii) We have

$$\check{p}_{01}(z) = \frac{c_0}{\alpha_1} \quad \text{for all } z \in \partial\mathbb{D} \quad (10.25)$$

with the constant  $c_0$  introduced in Lemma 18. (iv) We have

$$\frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{\partial}{\partial n} \check{p}_{01} ds = -1 \quad (10.26)$$

because of the definition of  $\alpha_1$  in (10.15). The normal derivative  $\partial/\partial n$  on  $\partial\mathbb{D}$  in (10.26) is assumed to be inwardly oriented.

From these four properties together with (10.4), it follows that in  $\mathbb{D}$  the function  $\check{p}_{01}$  is identical with the equilibrium potential  $p_{01}$  of the condenser  $(K_{0,1}, \mathbb{D})$ . From the first equality in (10.23) together with (10.25), it then follows that

$$\frac{1}{\text{cap}(K_{0,1}, \partial\mathbb{D})} = \frac{c_0}{\alpha_1}. \quad (10.27)$$

Motivated by (10.23) and (10.27), we define

$$\tilde{\alpha}_1 := \frac{\text{cap}(\tilde{K}_{0,1}, \partial\mathbb{D})}{\text{cap}(K_{0,1}, \partial\mathbb{D})} \alpha_1 < \alpha_1, \quad (10.28)$$

where the inequality is a consequence of (10.22).

Next, we study the function

$$\check{g}_0(z) := \begin{cases} \tilde{\alpha}_1(\check{p}_{01} \circ \varphi_1)(z) & \text{for } z \in \overline{U}_1 \\ g_{D_0}(z, \infty) & \text{for } z \in \overline{\mathbb{C}} \setminus \overline{U}_1, \end{cases} \quad (10.29)$$

which is basically a modification of the Green function  $g_{D_0}(z, \infty)$  in the neighborhood  $U_1$  of  $K_1$ . The function is continuous in  $\mathbb{C}$  since both partial functions in (10.29) are equal to  $c_0$  on  $\partial U_1$ . Indeed, it follows from (10.28), (10.27), and the second equation in (10.23) that

$$g_{D_0}(z, \infty) = \check{g}_0(z) = c_0 \quad \text{for all } z \in \partial U_1. \quad (10.30)$$

The function  $\check{g}_0$  has the following four properties: (i) It is harmonic in  $\overline{\mathbb{C}} \setminus (\tilde{K}_0 \cup \partial U_1)$  because of the definitions made in (10.18). (ii) It has smooth normal derivatives from both sides of  $\partial U_1$ . (iii) We have  $\check{g}_0(z) = 0$  for all  $z \in \tilde{K}_0$  because of the first line in (10.4). (iv) Near infinity we have

$$\check{g}_0(z) = \log |z| + \log \frac{1}{\text{cap}(K_0)} + o(1) \quad \text{as } z \rightarrow \infty, \quad (10.31)$$

which follows from (10.29) and Lemma 32 in Subsection 11.3, further below.

From the definition of  $\check{g}_0$  in (10.29) together with the four properties of  $\check{g}_0$  that have just been listed and with the use of Lemma 36 in Subsection 11.3, we deduce that

$$g_{\tilde{D}_0}(z, \infty) = \check{g}_0(z) + \int g_{\tilde{D}_0}(z, x) d\sigma(x), \quad z \in \overline{\mathbb{C}}, \quad (10.32)$$

where  $\sigma$  is a signed measure with  $\text{supp}(\sigma) \subset \partial U_1$ . From (11.57) in Lemma 36, we know that the measure  $\sigma$  is defined as the difference of the flux in  $\check{g}_0$  that comes to the Jordan curve  $\partial U_1$  from the both sites. Indeed, the total flux flowing into the set  $\overline{U}_1$  from outside is equal to  $\alpha_1$  because of the definition of  $\alpha_1$  in (10.15). On the other hand, the flux coming from within  $U_1$  is equal to  $\tilde{\alpha}_1$  because of (10.8) and (10.29). Hence, from (10.28) we conclude that

$$\sigma(\partial U_1) = \alpha_1 - \tilde{\alpha}_1 > 0. \quad (10.33)$$

Putting all partial results of the last paragraphs together, we arrive at the following estimate:

$$\begin{aligned} \log \text{cap}(K_0) - \log \text{cap}(\tilde{K}_0) &= (g_{\tilde{D}_0}(z, \infty) - \check{g}_0(z))|_{z=\infty} \\ &= \int g_{\tilde{D}_0}(x, \infty) d\sigma(x) \\ &= \int \check{g}_0(x) d\sigma(x) + \int g_{\tilde{D}_0}(x, y) d\sigma(x) d\sigma(y) \\ &> c_0(\alpha_1 - \tilde{\alpha}_1) > 0. \end{aligned} \quad (10.34)$$

Indeed, the first equality in (10.34) follows from (10.31) and representation (11.45) in Lemma 32 in Subsection 11.3, further below. The second one is a consequence of (10.32) and the symmetry of the Green function with respect to both of its arguments. The third one follows again from (10.32). The first inequality in (10.34) is a consequence of (10.30) and (10.33) together with the positive definiteness of the Green kernel (cf. Lemma 35 in Subsection 11.3, further below). The last inequality follows again from (10.33).

With the inequalities in (10.34) we have proved (10.20). It has already been mentioned after (10.20) that the proof of assertion (i) is complete as soon as we have completed the deduction of (10.19) and (10.20).

The considerations made in (10.29) with respect to the function  $\check{g}_0$  show that if each set  $K_{0,j}$ ,  $j = 1, \dots, m$ , is the unique solution of Problem  $(A_j, \mathbb{D})$ , therefore, identity (10.16) holds true for each  $j = 1, \dots, m$ . Hence, assertion (ii) is a consequence of assertion (i), and the proof of the whole Proposition 8 is complete.  $\square$

**10.1.3. The Minimal Set for Algebraic Functions.** With Proposition 8 and Theorem 14 we are prepared to prove a detailed description of the minimal set  $K_0(f, \infty)$  for Problem  $(f, \infty)$  with an algebraic function  $f$ .

The next theorem covers most of the content in the main theorems in Section 4 and 5. Since  $f$  is assumed to be an algebraic function, we deal here only with a special version of Problem  $(f, \infty)$ , however, we remark that at the present point the results of Section 4 and 5 are still not proved, and more than that, the results in the next theorem will later be used as intermediate steps in the general proofs.



THEOREM 15. *Let  $f$  be algebraic function that is not rational. We assume that the function is meromorphic at infinity. Let further  $K_0(f, \infty)$  be the minimal set for Problem  $(f, \infty)$ .*

(a) *The interior of  $K_0(f, \infty)$  is empty, and there exist two finite sets  $E_0, E_1$ , and a finite family of open and analytic Jordan arcs  $J_j, j \in I$ , such that*

$$K_0(f, \infty) = E_0 \cup E_1 \cup \bigcup_{j \in I} J_j. \quad (10.35)$$

*The components in (10.35) correspond to those in Theorem 4 of Section 4, but under the additional assumption that  $f$  is algebraic we can give a more specific characterization:*

- (i) *The set  $E_0$  is finite, and it consists of all branch points of  $f$  in  $K_0(f, \infty)$  that can be reached by meromorphic continuation of  $f$  out of the extremal domain  $D_0(f, \infty)$ .*
- (ii) *The set  $E_1$  is finite, and it consists of all bifurcation points of  $K_0(f, \infty)$  that do not belong to  $E_0$ .*
- (iii) *The family  $\{J_j\}_{j \in I}$  of analytic Jordan arcs is finite. All arcs  $J_j, j \in I$ , are pair-wise disjoint. The function  $f$  has meromorphic continuations across each arc  $J_j, j \in I$ , from both sides. Each arc  $J_j, j \in I$ , is a trajectory of the quadratic differential (10.36) having end points that belong to  $E_0 \cup E_1$ , and all open trajectories of (10.36) starting and ending at a point of  $E_0 \cup E_1$  belong to the family  $\{J_j\}_{j \in I}$ .*

(b) *The set  $E_0$  contains at least 2 points; we denote the points in  $E_0$  by  $a_1, \dots, a_n$ . There exist  $n - 2$  points  $b_1, \dots, b_{n-2} \in \mathbb{C}$  such that the Jordan arcs  $J_j, j \in I$ , are trajectories of the quadratic differential*

$$q(z) dz^2 \quad \text{with} \quad q(z) := \frac{(z - b_1) \dots (z - b_{n-2})}{(z - a_1) \dots (z - a_n)} \quad (10.36)$$

*Not all points of the set  $B = \{b_1, \dots, b_{n-2}\}$  are necessarily different, and not all of them are necessarily contained in  $K_0(f, \infty)$ .*

(c) *The minimal set  $K_0(f, \infty)$  consists of finitely many components; we denote their number by  $m$ . Each of these components contains at least two elements of  $E_0$ . We have  $E_1 \subset B$ . If  $m > 1$ , then the Green function  $g_{D_0}(\cdot, \infty)$ ,  $D_0 = D_0(f, \infty)$ , possesses critical points of total order  $m - 1$ , and each of these critical points appears in the set  $B$  with a frequency of twice its order.*

REMARK 4. *The Examples 6.1 - 6.4 in Section 6 belong to the class of problems covered by Theorem 15. In the discussion of these examples one finds concrete and explicit examples for the sets  $E_0, E_1$ , for the families of Jordan arcs  $\{J_j\}_{j \in I}$ , and also for the quadratic differentials (10.36).*

*There exists strong similarities between the two Theorems 15 and 9, but we note that the later one has a somewhat different orientation; it is focused only on the finiteness of the set  $E_0$ .*

PROOF. We define

$$q(z) := \left( \frac{\partial}{\partial z} g_{D_0}(z, \infty) \right)^2 \quad \text{for} \quad z \in D_0 = D_0(f, \infty) \quad (10.37)$$

with  $D_0(f, \infty)$  the extremal domain for Problem  $(f, \infty)$ . It is immediate that  $q$  is analytic in  $D_0(f, \infty)$ . From (10.37) and representation (11.45) for the Green function in Lemma 32 of Subsection 11.3, further below, we deduce that at infinity the function  $q$  has the development

$$q(z) = z^{-2} + O(z^{-3}) \quad \text{as } z \rightarrow \infty. \quad (10.38)$$

The function  $q$  is different from zero everywhere in  $D_0(f, \infty) \cap \mathbb{C}$  except at the critical points of the Green function  $g_{D_0}(\cdot, \infty)$ , where it has zeros. It follows from (10.37) that the order of each of these zeros is twice the order of the critical point. Critical points and their order have been introduced in Definition 7 in Subsection 5.3.

From Lemma 17 we know that  $K_0(f, \infty)$  has only a finite number of components, which we denote again by  $K_j$ ,  $j = 1, \dots, m$ . A combination of Proposition 8 and Theorem 14 shows that  $q$  is meromorphic in a neighborhood of each component  $K_j$ ,  $j = 1, \dots, m$ . Hence, the function is meromorphic throughout  $\overline{\mathbb{C}}$ , and consequently it is a rational function with all its poles contained in  $K_0(f, \infty)$ .

For the deduction of more specific assertions we can without loss of generality restrict our attention to individual components  $K_j$  and open neighborhoods  $U_j$ ,  $j = 1, \dots, m$ , of these sets. Without loss of generality we will choose  $j = 1$  in the sequel.

It follows from (10.16) and (10.15) in Proposition 8 together with assertion (ii) and (iii) of Theorem 14 that all poles of  $q$  on  $K_1$  are simple, and they have to belong to the set  $E_1$  from (10.11), i.e., they have to be branch points of  $f$  on  $K_1$ .

Further it follows especially from assertion (ii) of Theorem 14 that on  $K_1$  the function  $q$  has exactly two zeros less than it has poles on  $K_1$ , where multiplicities of zeros have to be taken into account. The zeros in question constitute the set  $E_1 \cap K_1$ .

In the application of assertion (ii) of Theorem 14 there may appear cancellations of numerator and denominator factors in the function (10.5). If none of such cancellations occurs, which can be seen as the generic case, then every branch point of the function  $f$  on  $K_1$  corresponds to a simple pole of the function  $q$  on  $K_1$ .

From what has been proved so far together with the definitions made in (10.15) and the identity (10.16) of Proposition 8, we deduce from assertion (ii) of Theorem 14 that all Jordan arcs  $J_j$ ,  $j \in I$ , that belong to  $K_1$  are transformed by the conformal map  $\varphi_1$  of (10.15) into a critical trajectory  $\varphi_1(J_j)$  of the quadratic differential (10.5) of Theorem 14, and the reverse conclusion holds also true. With these last conclusions we have proved assertion (iii) of part (a) in the theorem for the component  $K_1$ .

All conclusions that have been proved so far for the component  $K_1$  hold true in the same way on the other  $m - 1$  components  $K_j$ ,  $j = 2, \dots, m$ , of the minimal set  $K_0(f, \infty)$ , which proves practically all assertions of the theorem.

We add that the number of zeros of the rational function  $q$  on all  $m$  component  $K_1, \dots, K_m$  together with the  $2(m - 1)$  zeros at the critical points of the Green function  $g_{D_0}(\cdot, \infty)$  add up to exactly two zeros less than the number of poles that  $q$  has in  $\mathbb{C}$ . This account reaffirms exactly the behavior of the function  $q$  at infinity, which is shown in development (10.38).  $\square$

**10.2. Some Technical Results.** In the present subsection we prove some technical results which then are needed in the remainder of the section in

proofs of results from the Sections 4, 5, and 7. Most important are here the proofs of the two Theorems 4 and 11. In a first part of the subsection, the results will be formulated together with related definitions; proofs will then follow afterwards. Some of the results depend on rather subtle topological assumptions.

The first proposition is especially important for the proof of Theorem 11.

**PROPOSITION 10.** *Let  $D_1, D_2 \in \mathcal{D}(f, \infty)$  be two admissible domains for Problem  $(f, \infty)$ . If we assume that  $D_1$  possesses the  $S$ -property as introduced in Definition 9, and if we assume further that  $D_2$  is elementarily maximal in the sense of Definition 8, then we have either  $D_1 = D_2$  or*

$$\text{cap}(\partial D_1) < \text{cap}(\partial D_2). \quad (10.39)$$

Besides of Proposition 10 we need a very similar result, which is not related to admissible domains  $D \in \mathcal{D}(f, \infty)$ ; instead it is based on purely topological assumptions, which however comes to the same thing. We prepare the formulation of the result by some definitions.

**DEFINITION 19.** *Let  $K, E \subset \mathbb{C}$  be two polynomials-convex and compact sets with  $\text{cap}(K) > 0$  and  $E \subset K$ . We say that the set  $K$  possesses the  $S$ -property on the subset  $K \setminus E$  if the following two assertions are satisfied:*

(i) *The set  $K \setminus E$  is of the form*

$$K \setminus E = E_1 \cup \bigcup_{j \in I} J_j \quad (10.40)$$

*with  $E_1$  a discrete set in  $K \setminus E$  and  $\{J_j\}_{j \in I}$  a family of smooth, open, and disjoint Jordan arcs  $J_j$ . Each point  $z \in E_1$  is an end point of at least three different arcs from  $\{J_j\}_{j \in I}$ .*

(ii) *The Green function  $g_D(\cdot, \infty)$  with  $D := \overline{\mathbb{C}} \setminus K$  satisfies the symmetry relation*

$$\frac{\partial}{\partial n_+} g_D(z, \infty) = \frac{\partial}{\partial n_-} g_D(z, \infty) \quad \text{for all } z \in J_j, j \in I \quad (10.41)$$

*with  $\partial/\partial n_+$  and  $\partial/\partial n_-$  denoting the normal derivatives to both sides of the arcs  $J_j$ ,  $j \in I$ .*

It is obvious that there exist many parallels between the Definitions 19 and Definitions 9 of Section 7; the special aspect of the new definition is the independence from Problem  $(f, \infty)$  and the admissible domains  $D \in \mathcal{D}(f, \infty)$ . On the other hand, it is immediate that any admissible domain  $D \in \mathcal{D}(f, \infty)$  with  $K := \overline{\mathbb{C}} \setminus D$  that possesses the  $S$ -property in the sense of Definition 9 possesses also the  $S$ -property in the sense of Definition 19 on the subset  $K \setminus E_0$ , where  $E_0$  is the compact set from (7.1) and assertion (i) in Definition 9.

Let  $K_1, K_2, E \subset \mathbb{C}$  be three compact sets with  $E \subset K_1 \cap K_2$ . Two components  $E_1, E_2 \subset E$  of  $E$  are said to be connected in  $K_1$  if both components are contained in the same component of  $K_1$ . In this sense,  $K_1$  defines a connectivity relation on the components of  $E$ . We say (in the usual sense) that the connectivity of  $E$  in  $K_1$  is coarser than the connectivity in  $K_2$  if the connectedness of two components  $E_1, E_2 \subset E$  in  $K_2$  implies their connectedness in  $K_1$ .

DEFINITION 20. Let  $K_1, K_2, E \subset \mathbb{C}$  be three polynomial-convex and compact sets with  $\text{cap}(K_1) > 0$  and  $E \subset K_1 \cap K_2$ , and let us assume that  $K_1$  possesses the  $S$ -property in the sense of Definition 19 on  $K_1 \setminus E$  with  $\{J_j\}_{j \in I}$  denoting the family of Jordan arcs introduced in (10.40). We say that the connectivity of  $E$  in  $K_1$  is minimally coarser than the connectivity of  $E$  in  $K_2$  if the following two assertions hold true:

- (i) The connectivity of  $E$  in  $K_1$  is coarser than that in  $K_2$ .
- (ii) If  $\tilde{K}_1$  is the compact set that results from dropping one of the arcs  $J_j$ ,  $j \in I$ , from  $K_1$ , then assertion (i) holds no longer true with  $\tilde{K}_1$  replacing  $K_1$ .

REMARK 5. Since the arcs  $J_j$ ,  $j \in I$ , in (10.40) are assumed to be open, one can drop any arc  $J_{j_0}$ ,  $j_0 \in I$ , from  $K$ , and the remaining set  $\tilde{K} = K \setminus J_{j_0}$  is still compact and polynomial-convex, but of course, the connectivity defined by  $\tilde{K}$  is finer than that defined by  $K$ .

PROPOSITION 11. Let  $K_1, K_2, E \subset \mathbb{C}$  be three polynomial-convex and compact sets with  $\text{cap}(K_j) > 0$ ,  $j = 1, 2$ , and  $E \subset K_1 \cap K_2$ . Let us assume further that  $K_1$  possesses the  $S$ -property in the sense of Definition 19 on  $K_1 \setminus E$  and that the connectivity of  $E$  in  $K_1$  is minimally coarser than the connectivity in  $K_2$ . Then at least one of the two assertions

$$K_1 \subset K_2 \quad \text{or} \quad \text{cap}(K_1) < \text{cap}(K_2) \quad (10.42)$$

holds true.

The proof of Proposition 11 will follow in the footsteps of Proposition 10 only that at its beginning there are differences because of the different type of assumptions in Proposition 10.

In the next proposition a bridge is built between the set-up of Proposition 11 and the world of Problem  $(f, \infty)$  with its admissible domains  $D \in \mathcal{D}(f, \infty)$ . The assumptions of the proposition are rather technical, but they are constructed in such a way that they fit well to the situation in the proof of Theorem 4, further below, where Proposition 11 is needed in a crucial way.

PROPOSITION 12. Let the admissible domain  $D \in \mathcal{D}(f, \infty)$  be elementarily maximal in the sense of Definition 8 with  $\text{cap}(\partial D) > 0$ . Set  $K := \overline{\mathbb{C}} \setminus D$ , and let  $E_0 \subset K$  denote the minimal compact and polynomial-convex set with the property that  $\partial E_0$  contains all points for which assertion (i) of Definition 8 holds true.

Let  $U \subset \mathbb{C}$  be an open set with  $E_0 \subset U$ , set  $E := \overline{U} \cap K$ , and assumed that there exists a polynomial-convex and compact set  $K_1 \subset \mathbb{C}$  satisfying the following assertions:

- (i)  $\text{cap}(K_1) > 0$  and  $E \subset K_1$ .
- (ii)  $K_1$  possesses the  $S$ -property in the sense of Definition 19 on  $K_1 \setminus E$ .
- (iii) The connectivity of  $E$  in  $K_1$  is minimally coarser than the connectivity of  $E$  in  $K$ .
- (iv) We have  $K_1 \setminus K \neq \emptyset$ .

If these assumptions are satisfied, then there exists an admissible domain  $\tilde{D} \in \mathcal{D}(f, \infty)$  with

$$\text{cap}(\partial\tilde{D}) < \text{cap}(\partial D). \quad (10.43)$$

We now come to a proposition, which will play a technical role in the proof of Theorem 4.

**PROPOSITION 13.** *Let  $K, E \subset \mathbb{C}$  be two polynomial-convex and compact sets with  $\text{cap}(K) > 0$  and  $E \subset K$ . We assumed that  $K$  possesses the  $S$ -property in the sense of Definition 19 on  $K \setminus E$ , and by  $E_1 \subset K \setminus E$  and  $\{J_j\}_{j \in I}$  we denote the compact discrete set and the family of open Jordan arcs introduced in (10.40). We set  $D := \overline{\mathbb{C}} \setminus K$  and defined the function  $q$  by*

$$q(z) := \left( 2 \frac{\partial}{\partial z} g_D(z, \infty) \right)^2 \quad \text{for } z \in \overline{\mathbb{C}} \setminus E \quad (10.44)$$

with  $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$  the usual complex differentiation and  $g_D(\cdot, \infty)$  the Green function the domain  $D$ .

The function  $q$  is analytic in  $\overline{\mathbb{C}} \setminus E$  as a consequence of the assumed  $S$ -property of  $K$  on  $K \setminus E$ , it has a zero at each point  $z \in E_1$ , the order of each of these zeros  $z \in E_1$  is equal to the number of different arcs from  $\{J_j\}_{j \in I}$  that have  $z$  as their end point minus 2, i.e., it is of order  $i(z) - 2$  with  $i(z)$  denoting the bifurcations index introduced in Definition 6, the function  $q$  is different from zero in  $\overline{\mathbb{C}} \setminus (E \cup E_1)$  except at the critical points of  $g_D(\cdot, \infty)$  (cf. Definition 7) and at infinity, further we have the estimate

$$|q(z)| \leq \frac{3}{\text{dist}(z, E)^2} \left( \log(3r) + \log \frac{1}{\text{cap}(K)} \right) \quad \text{for all } z \in \{|z| \leq r\} \setminus E \quad (10.45)$$

and any  $r > 0$  sufficiently large so that  $K \subset \{|z| \leq r\}$ .

The most important part of Proposition 13 is the estimate (10.45). We underlined that this estimate depends only on  $\text{cap}(K)$  and the set  $E$ , but not on the shape or extension of the set  $K$  or the complementary domain  $D$ .

It has been stated in Proposition 13 that the  $S$ -property of a compact set  $K$  on  $K \setminus E$  implies the analyticity of the function  $q$  defined in (10.44), is a rather immediate conclusion of (10.41) and (10.44). It is interesting that also the reverse conclusion holds true, which is formulated in the next lemma.

**LEMMA 19.** *Let the function  $q$  be defined by (10.44) with the same notations as those introduced and use in Proposition 13, and assume further that  $q$  is analytic in  $\overline{\mathbb{C}} \setminus E$ . Then the set  $K$  possesses the  $S$ -property on  $K \setminus E$  in the sense of Definition 19.*

We next come to the proofs of the four Propositions 10 - 13 and of Lemma 19.

10.2.1. *Proof of Proposition 10.* The proof of Proposition 10 is rather involved and as an essential tool the Dirichlet integral of a Green function is used.

PROOF OF PROPOSITION 10. We assume that

$$D_1 \neq D_2, \quad (10.46)$$

and show then that this implies (10.39).

Set  $K_j := \overline{\mathbb{C}} \setminus D_j$ ,  $j = 1, 2$ , and denote by  $\tilde{E}_{0,j} \subset K_j$ ,  $j = 1, 2$ , the two sets of points  $z \in K_j$  for which assertion (i) in Definition 9 and in Definition 8 are satisfied for the domains  $D_1$  and  $D_2$ , respectively. Let  $E_{0,j}$  be the polynomial-complex hull of  $\tilde{E}_{0,j}$ , i.e.,

$$E_{0,j} := \widehat{\tilde{E}_{0,j}}, \quad j = 1, 2. \quad (10.47)$$

Further, we defined

$$K_3 := \widehat{K_1 \cup K_2} \quad \text{and} \quad D_3 := \overline{\mathbb{C}} \setminus K_3. \quad (10.48)$$

Since the domain  $D_1$  has been assumed to possess the  $S$ -property, we know from (7.1) in Definition 9 that  $K_1$  can be represented in the form

$$K_1 = E_{1,0} \cup E_1 \cup \bigcup_{j \in I} J_j \quad (10.49)$$

with the two sets  $E_{1,0}$ ,  $E_1$ , and the family of Jordan arcs  $J_j$ ,  $j \in I$ , with properties as described in Definition 9.

In the next step we study some properties of the two sets  $K_1 \setminus K_2$  and  $K_2 \setminus K_1$  that follow immediately from assumption (10.46). We have

$$(\partial K_3 \setminus K_2) \cap E_{0,1} = \emptyset \quad \text{and} \quad (\partial K_3 \setminus K_1) \cap E_{0,2} = \emptyset. \quad (10.50)$$

Indeed, from (10.48) it follows that  $\partial K_3 \subset \partial K_1 \cup \partial K_2$ . Since  $D_3 \subset D_1 \cap D_2$ , and since both domains  $D_1$  and  $D_2$  are elementally maximal,  $z \in \tilde{E}_{0,1} \cap \partial K_3$  implies  $z \in \tilde{E}_{0,2} \cap \partial K_3$ , and vice versa. Consequently, we have

$$E_{0,j} \cap \partial K_3 \subset K_1 \cap K_2 \cap \partial K_3 \quad \text{for} \quad j = 1, 2, \quad (10.51)$$

which proves (10.50).

We have

$$\text{cap}(K_2 \setminus K_1) > 0 \quad \text{and} \quad \text{cap}(K_1 \setminus K_2) > 0. \quad (10.52)$$

Here, we first prove that  $\text{cap}(K_2 \setminus K_1) > 0$ . This will be done in an indirect way, and we assume for this purpose that

$$\text{cap}(K_2 \setminus K_1) = 0. \quad (10.53)$$

Let  $f_j$  denote the meromorphic continuations of the function  $f$  into the domains  $D_j$ ,  $j = 1, 2, 3$ . With the same arguments as used in the proof of Lemma 4 in Subsection 9.2, we can show that assumption (10.53) implies that all meromorphic continuations of the function  $f_2$  out of  $D_1 \setminus K_2$  into  $D_1$  lead to the same function in each component of the open set  $D_1 \setminus K_2$ . These functions then are necessarily identical with the function  $f_1$ . Since the domain  $D_2$  has been assumed to be elementarily maximal, it follows from assertion (ii) in Definition 8 that  $K_2 \setminus K_1 = \emptyset$ , which implies that  $K_2 \subset K_1$ . As a consequence of the assumed  $S$ -property of the domain  $D_1$ , we know that  $D_1$  is also elementarily maximal (see the assertions (i) and (ii) in Definition 9), and therefore  $D_2 \supset D_1$  implies that  $D_1 = D_2$ . This last conclusion contradicts (10.46), and consequently we have proved  $\text{cap}(K_2 \setminus K_1) > 0$

in (10.52). A proof of  $\text{cap}(K_1 \setminus K_2) > 0$  in (10.52) can be done in exactly the same way.

From (10.49) and the fact that the two sets  $\partial K_3 \setminus K_2$  and  $E_{0,1}$  are disjoint, which has been proved in (10.50), we immediately conclude that

$$\partial K_3 \setminus K_2 \subset E_1 \cup \bigcup_{j \in I} J_j \quad (10.54)$$

i.e.,  $\partial K_3 \setminus K_2$  is the union of open subarcs of arcs from the family  $\{J_j\}_{j \in I}$  together with points from  $E_1$ . The dominant parts in this union are the open Jordan arcs since the set  $E_1$  is countable, and therefore we have  $\text{cap}(E_1) = 0$ .

We now continue our investigation with further definitions. We set

$$V := \overline{\text{Int}(K_3) \cap (K_1 \setminus K_2)}, \quad \tilde{D}_2 := D_2 \setminus V, \quad \tilde{K}_2 := K_2 \cup V = \mathbb{C} \setminus \tilde{D}_2. \quad (10.55)$$

It is immediate that  $\tilde{D}_2$  is open, but it is not necessarily a domain. By  $g_j(\cdot, \cdot)$  we denote the Green functions  $g_{D_j}(\cdot, \cdot)$  in the domains  $D_j$ ,  $j = 1, 2$ . Because of (10.52), these two Green functions exist in a proper sense (see Subsection 11.3, further below).

Next, we show that

$$K_1 \cap \tilde{D}_2 \subset \bigcup_{j \in I} J_j \cap \partial \text{Int}(K_3), \quad (10.56)$$

or more precisely, we show that  $K_1 \cap \tilde{D}_2$  consists only of open subarcs of the arcs  $J_j$  from (10.54) that are contained in  $\partial \text{Int}(K_3)$ . Indeed, since  $K_1$  possesses the  $S$ -property of Definition 9, it follows from assertion (ii) in Definition 9 that  $K_1 \subset \overline{\text{Int}(K_3)}$ . It further follows from the definitions in (10.55) that  $K_1 \cap \tilde{D}_2 \subset \partial K_3 \setminus K_2$ . Because of (10.54), it remains only to show that  $K_1 \cap \tilde{D}_2 \cap E_1 = \emptyset$ . Let us assume that  $z \in K_1 \cap \tilde{D}_2 \cap E_1$ . From assertion (iv) in Definition 9 of the  $S$ -property we know that at least three different arcs of the family  $\{J_j\}$  in (10.49) have  $z$  as endpoint. Since  $K_1 \cap \tilde{D}_2$  lies in  $\partial K_3 \setminus K_2$ , the meromorphic continuations of the two functions  $f_1$  and  $f_2$  out of the domain  $D_3$  are identical, and therefore, at least one of the arcs ending at  $z$  belongs to  $V$ ; and consequently, we have  $z \in V$ , which contradicts  $z \in K_1 \cap \tilde{D}_2$ . Thus, (10.56) is proved.

A key role in the proof of the proposition is played by the function  $\tilde{g}_1$ , which is defined as

$$\tilde{g}_1(z) := \begin{cases} g_1(z, \infty) & \text{for } z \in D_3, \\ -g_1(z, \infty) & \text{for } z \in K_3 \setminus K_2. \end{cases} \quad (10.57)$$

All discussions, so far, can be seen as preliminaries to an investigation of properties of the function  $\tilde{g}_1$ . In this connection the  $S$ -property of the domain  $D_1$  is very important since it implies that the two pieces in the definition of the function  $\tilde{g}_1$  are harmonic continuations of each other across the arcs in  $K_1 \cap \tilde{D}_2$ .

Indeed, from symmetry (7.2) in Definition 9 together with (10.56) and the remarks just after (10.56), we conclude that  $\tilde{g}_1$  is harmonic in  $\tilde{D}_2$ . Notice that the domain  $D_1$  is assumed to possess the  $S$ -property.

The function  $\tilde{g}_1$  is superharmonic in  $D_2$ . Indeed, from Lemma 32 in Subsection 11.3, further below, we know that the Green function  $g_1(z, \infty)$  is subharmonic in  $\mathbb{C}$ . Since  $V \subset K_3 \setminus K_2$ , the superharmonicity follows directly from (10.57).

From the defining identity (11.43) of the Green function in Subsection 11.3, further below, together with (10.57) and the definition of  $V$  in (10.55), we conclude that

$$\tilde{g}_1(z) = 0 \quad \text{for quasi every } z \in V. \quad (10.58)$$

From the properties of  $\tilde{g}_1$  that have just been discussed together with the Poisson-Jensen Formula (cf. Theorem 18 in Subsection 11.3, further below) we get for  $\tilde{g}_1$  the representation

$$\tilde{g}_1(z) = \tilde{h}_1(z) + g_2(z, \infty) + g_0(z) \quad \text{for } z \in D_2, \quad (10.59)$$

where  $g_2(\cdot, \infty)$  is the Green function in  $D_2$ , and  $g_0$  the Green potential

$$g_0(z) = \int_V g_2(z, v) d\omega_1(v) \quad (10.60)$$

with  $\omega_1$  the equilibrium distribution on  $K_1$ , and  $V$  the set from (10.55). The function  $\tilde{h}_1$  in (10.59) is the solution of the Dirichlet problem in  $D_2$  with boundary values

$$\tilde{h}_1(z) = \tilde{g}_1(z) \quad \text{for quasi every } z \in \partial K_2. \quad (10.61)$$

Identity (10.59) can easily be verified by considering its values on  $\partial D_2$ , on  $V$ , and near infinity.

From (10.52) and Lemma 33 in Subsection 11.3, further below, we deduce that the two Green functions  $g_1(\cdot, \infty)$  and  $g_2(\cdot, \infty)$  are essentially different, and consequently, we have

$$g_1(z, \infty) > 0 \quad \text{for quasi every } z \in K_2 \setminus K_1. \quad (10.62)$$

From (10.61) and (10.57), we then conclude that

$$\tilde{h}_1 \neq 0. \quad (10.63)$$

By definition, we have  $g_0(z) \geq 0$  for all  $z \in \overline{D_2}$ , and  $g_0(z) > 0$  for  $z \in D_2$  if, and only if,  $\omega_1(V) > 0$ . However, this last condition may in general not be satisfied; even the case  $V = \emptyset$  is possible.

In order to prove (10.39), we prove that the identity

$$\begin{aligned} \log \frac{\text{cap}(K_2)}{\text{cap}(K_1)} &= D_{K_2 \setminus K_1}(g_1(\cdot, \infty)) + D_{D_2}(\tilde{h}_1) + 2g_0(\infty) \\ &\quad + \int_V \int_V g_2(v, w) d\omega_1(v) d\omega_1(w) \end{aligned} \quad (10.64)$$

holds true, where  $D_{K_2 \setminus K_1}(g_1(\cdot, \infty))$  and  $D_{D_2}(\tilde{h}_1)$  are Dirichlet integrals that have been introduced in (11.56) in Subsection 11.3, further below.

Since we know from (10.63) that the harmonic function  $\tilde{h}_1$  is not identical zero in  $D_2$ , it follows from the definition of the Dirichlet integral in (11.56) that

$$D_{K_2 \setminus K_1}(g_1(\cdot, \infty)) \geq 0 \quad \text{and} \quad D_{D_2}(\tilde{h}_1) > 0. \quad (10.65)$$

We have already mentioned that the Green potential  $g_0$  is always non-negative. It follows from (10.60) and the positivity of the Green function as kernel function (see Lemma 35 in Subsection 11.3, further below) that

$$g_0(\infty) \geq 0, \quad \int_V \int_V g_2(v, w) d\omega_1(v) d\omega_1(w) \geq 0, \quad (10.66)$$

and we have proper inequalities in both cases of (10.66) if, and only if,  $\omega_1(V) > 0$ .



When identity (10.64) is proved, then inequality (10.39) follows immediately from identity (10.64) together with (10.65) and (10.66). Hence, it remains only to prove that (10.64) holds true, which will be done next.

From Lemma 32, Lemma 37, and Corollary 3 in Subsection 11.3, further below, we deduce that

$$\begin{aligned} \log \frac{\text{cap}(K_2)}{\text{cap}(K_1)} &= (g_1(\cdot, \infty) - g_2(\cdot, \infty))(\infty) \\ &= D_{D_{1,r}}(g_1(\cdot, \infty)) - D_{D_{2,r}}(g_2(\cdot, \infty)) + O\left(\frac{1}{r}\right) \end{aligned} \quad (10.67)$$

as  $r \rightarrow \infty$ , where  $D_{j,r}$  denotes the bounded domain

$$D_{j,r} := D_j \cap \{|z| < r\} \quad \text{for } j = 1, 2, \quad (10.68)$$

and  $r > 0$  so large that  $K_j \subset \{|z| < r\}$  for  $j = 1, 2, 3$ .

From the definition of the Dirichlet integral in (11.56), further below, and the definition of the function  $\tilde{g}_1$  in (10.57), we deduce that

$$\begin{aligned} D_{D_{1,r}}(g_1(\cdot, \infty)) &= D_{K_2 \setminus K_1}(g_1(\cdot, \infty)) + D_{\tilde{D}_{2,r}}(g_1(\cdot, \infty)) \\ &\quad D_{K_2 \setminus K_1}(g_1(\cdot, \infty)) + D_{\tilde{D}_{2,r}}(\tilde{g}_1). \end{aligned} \quad (10.69)$$

In (10.69), the open set  $\tilde{D}_{2,r}$  is defined as  $\tilde{D}_{2,r} := \tilde{D}_2 \cap \{|z| < r\}$  in analogy to (10.68). It has already been mentioned in (10.65) that  $D_{K_2 \setminus K_1}(g_1(\cdot, \infty)) \geq 0$ .

We will now have a closer look on the Dirichlet integral  $D_{\tilde{D}_{2,r}}(\tilde{g}_1)$  in (10.69). From the representations (10.59), (10.60), and equality (10.58), it follows with the help of Lemma 40 in Subsection 11.3, further below, that

$$D_{\tilde{D}_{2,r}}(\tilde{g}_1) = D_{D_{2,r}}(\tilde{h}_1 + g_2(\cdot, \infty) + g_{0,r}) + \int_V \int_V g_2(v, w) d\omega_1(v) d\omega_1(w) \quad (10.70)$$

with  $\omega_1$  and  $V$  defined like in (10.60). In (10.70), the function  $g_{0,r}$  denotes the solution of the Dirichlet problem in  $D_{2,r}$  with boundary values  $g_{0,r}(z) = g_0(z)$  for quasi every  $z \in \partial D_{2,r}$ . We know therefore from (10.60) that

$$g_{0,r}(z) = \begin{cases} 0 & \text{for quasi every } z \in \partial D_2, \\ g_0(z) & \text{for } |z| = r. \end{cases} \quad (10.71)$$

Next, we investigate the Dirichlet integral  $D_{D_{2,r}}(\tilde{h}_1 + g_2(\cdot, \infty) + g_{0,r})$  in (10.70) in more detail. Using the notations introduced in (10.59), (10.70), and also in (11.55), further below, we prove the identity

$$\begin{aligned} D_{D_{2,r}}(\tilde{h}_1 + g_2(\cdot, \infty) + g_{0,r}) &= D_{D_{2,r}}(\tilde{h}_1) + D_{D_{2,r}}(g_2(\cdot, \infty)) + D_{D_{2,r}}(g_{0,r}) + \\ &\quad + 2 D_{D_{2,r}}(\tilde{h}_1, g_2(\cdot, \infty)) + 2 D_{D_{2,r}}(\tilde{h}_1, g_{0,r}) + 2 D_{D_{2,r}}(g_{0,r}, g_2(\cdot, \infty)). \end{aligned} \quad (10.72)$$

Indeed, the positivity of the integrand in the Dirichlet integral implies that

$$\lim_{r \rightarrow \infty} D_{D_{2,r}}(\tilde{h}_1) = D_{D_2}(\tilde{h}_1) > 0, \quad (10.73)$$

and since  $\tilde{h}_1$  is harmonic and bounded in  $D_2$ , the Dirichlet integral on the right-hand side of (10.73) is finite.

The Green potential  $g_0$  is bounded near infinity, therefore it follows from the definition of  $g_{0,r}$  as a solution of a Dirichlet problem in  $D_{2,r}$  with boundary values (10.71) that

$$\lim_{r \rightarrow \infty} g_{0,r}(z) = 0 \quad \text{locally uniformly for } z \in D_2, \quad (10.74)$$

and the same conclusion holds also for the first derivatives in  $\nabla g_{0,r}$  and  $r \rightarrow \infty$ ; these derivatives converge also to zero locally uniformly in  $D_2$ . As a consequence, we have

$$\lim_{r \rightarrow \infty} D_{D_{2,r}}(g_{0,r}) = 0. \quad (10.75)$$

With the Cauchy-Schwartz inequality and the boundedness of  $D_{D_{2,r}}(\tilde{h}_1)$  we deduce from (10.75) that we also have

$$\lim_{r \rightarrow \infty} D_{D_{2,r}}(\tilde{h}_1, g_{0,r}) = 0. \quad (10.76)$$

The function  $\tilde{h}_1$  is harmonic in  $D_2$ , and therefore it follows from Lemma 38 in Subsection 11.3, further below, that

$$\lim_{r \rightarrow \infty} D_{D_{2,r}}(\tilde{h}_1, g_2(\cdot, \infty)) = 0. \quad (10.77)$$

Since the Green potential  $g_0$  is harmonic in  $\{|z| > r\}$ , we deduce from (10.71) and Lemma 39 in Subsection 11.3, further below, that

$$D_{D_{2,r}}(g_{0,r}, g_2(\cdot, \infty)) = \frac{1}{2\pi} \int_{\{|z|=r\}} g_0(z) \frac{\partial}{\partial n} g_2(z, \infty) ds_z = g_0(\infty) \quad (10.78)$$

for  $r > 0$  sufficiently large.

From identity (10.72) together with (10.73), (10.75), (10.76), (10.77), and (10.78), we get

$$\lim_{r \rightarrow \infty} \left( D_{D_{2,r}}(\tilde{h}_1 + g_2(\cdot, \infty) + g_{0,r}) - D_{D_{2,r}}(g_2(\cdot, \infty)) \right) = D_{D_2}(\tilde{h}_1) + 2g_0(\infty). \quad (10.79)$$

Further, we then get from formula (10.67) together with (10.69), (10.70), and (10.79) that identity (10.64) holds true, which completes the proof of the proposition.  $\square$

**10.2.2. Proof of Proposition 11.** It has already been mentioned that the proof of Proposition 11 follows in the footsteps of that of Proposition 10 only that we have a modified opening since we have to start from a different type of assumptions. But after the introduction of the function  $\tilde{g}_1$  in (10.57), we can use the argumentation of the last proof without any change.

**PROOF OF PROPOSITION 11.** We assume

$$K_1 \not\subseteq K_2, \quad (10.80)$$

and show then that this implies  $\text{cap}(K_1) < \text{cap}(K_2)$ .

Like in the proof of Proposition 10, we set

$$K_3 := \widehat{K_1 \cup K_2} \quad \text{and} \quad D_j := \overline{\mathbb{C}} \setminus K_j, j = 1, 2, 3, \quad (10.81)$$

where  $\hat{\phantom{x}}$  denotes the polynomial-convex hull (cf. Definition 22 in Subsection 11.1, further below). Since it has been assumed that the compact set  $K_1$  possesses the  $S$ -property on  $K_1 \setminus E$  in the sense of Definition 19, in  $K_1$  we have the compact set

$E_1$  and the family of Jordan arcs  $\{J_j\}_{j \in I}$  that have been introduced in (10.40) of Definition 19, and from there we then know that

$$K_1 = E \cup E_1 \cup \bigcup_{j \in I} J_j. \quad (10.82)$$

From assumption (10.80), representation (10.82), assumption  $\text{cap}(K_1) > 0$ ,  $E \subset K_2$ , and the fact that the capacity of an open Jordan arc is positive (cf. Lemma 20 in Subsection 11.1, further below), it follows that

$$\text{cap}(K_1 \setminus K_2) > 0. \quad (10.83)$$

The boundary  $\partial C$  of any component  $C$  of the open set

$$O := \text{Int}(K_3) \setminus K_2 \quad (10.84)$$

contains elements of  $\partial K_1$  and  $\partial K_2$  because of the assumed polynomial-convexity of  $K_1$ . We have

$$(K_1 \setminus K_2) \cap \partial O \subset K_1 \setminus E, \quad (10.85)$$

and if we define the set  $V$  by

$$V := \overline{\text{Int}(K_3) \cap (K_1 \setminus K_2)}, \quad (10.86)$$

then it follows from (10.82), (10.85), and (10.86) that

$$(K_1 \setminus (K_2 \cup V)) \cap \partial O \subset \bigcup_{j \in I} J_j. \quad (10.87)$$

On the other hand, we have

$$K_1 \setminus (K_2 \cup V) \subset \partial O, \quad (10.88)$$

since otherwise any arc  $J_j$ ,  $j \in I$ , in  $K_1 \setminus (K_2 \cup V \cup \partial O)$  could be removed from  $K_1$  without separating any pair of components of the set  $E$  in the modified set  $K_1$  if this pair is already connected in  $K_2$ . But such a situation would contradict the assumption that the connectivity of the set  $E$  in  $K_1$  is minimal coarser than the connectivity in  $K_2$  (cf. Definition 20).

Like in (10.55) in the proof of Proposition 10, we define

$$\tilde{D}_2 := D_2 \setminus V \quad \text{and} \quad \tilde{K}_2 := K_2 \cup V = \mathbb{C} \setminus \tilde{D}_2 \quad (10.89)$$

with the compact set  $V$  introduced in (10.86). It follows from (10.87) and (10.88) that

$$K_1 \cap \tilde{D}_2 \subset \bigcup_{j \in I} J_j \cap \partial \text{Int}(K_3). \quad (10.90)$$

Indeed, because of (10.88) we have  $K_1 \cap \tilde{D}_2 = K_1 \setminus (K_2 \cup V) \subset \partial O \setminus K_2 \subset \partial \text{Int}(K_3)$ , and because of (10.87) together with (10.88) we have  $K_1 \cap \tilde{D}_2 = K_1 \setminus (K_2 \cup V) \subset \bigcup_{j \in I} J_j$ .

After these preparations we can now follow the argumentation in the proof of Proposition 10 word-for-word. Exactly, like in (10.57) we define the function  $\tilde{g}_1$  in the domain  $D_2$ . With the same arguments as those used in the proof of Proposition 10 after (10.57) we then arrive after many intermediate steps at the conclusion that

$$\text{cap}(K_1) < \text{cap}(K_2), \quad (10.91)$$

which proves Proposition 11.  $\square$

10.2.3. *Proof of Proposition 12.* The proof of Proposition 12 relies strongly on Definition 17 and the subsequent Theorem 13 together with Proposition 11.

PROOF OF PROPOSITION 12. We set

$$K_0 := K, \quad K_3 := \widehat{K_0 \cup K_1}, \quad \text{and} \quad D_j := \overline{\mathbb{C}} \setminus K_j, j = 0, 1, 3. \quad (10.92)$$

In (10.92),  $\widehat{\cdot}$  denotes the polynomial-convex hull (cf. Definition 22 in Subsection 11.1, further below). It is immediate that the  $D_j$  are domains.

From Proposition 11 together with the assumptions (i) - (iv) of the proposition, it follows that

$$\text{cap}(K_1) < \text{cap}(K_0). \quad (10.93)$$

If we would know that  $D_1 = \overline{\mathbb{C}} \setminus K_1 \in \mathcal{D}(f, \infty)$ , then the proof of the proposition would be completed with (10.93). However, in general we have  $D_1 \notin \mathcal{D}(f, \infty)$  since the meromorphic continuation of the function  $f$  out of the domain  $D_3$  into  $D_1$  may hit non-polar similarities in  $D_1 \cap \text{Int}(K_3)$ .

In order to overcome these difficulties we make use of a construction introduced in Definition 17. We consider the whole family of compact sets  $K_h$ ,  $h \in [0, 1]$ , with complementary domains  $D_h = \overline{\mathbb{C}} \setminus K_h$  that are defined by the relations (9.139) through (9.145) of Definition 17 starting from the two domains  $D_0$  and  $D_1$  defined in (10.92), and with elements of the proof of Theorem 13 we then prove that there exist  $h_0 \in (0, 1)$  such that

$$D_h \in \mathcal{D}(f, \infty) \quad \text{for all } 0 < h \leq h_0, \quad (10.94)$$

and further that

$$\log \text{cap}(K_h) < (1 - h) \log \text{cap}(K_0) + h \log \text{cap}(K_1) \quad \text{for } 0 < h < 1. \quad (10.95)$$

From (10.95) we deduce that

$$\log \text{cap}(K_{h_0}) < \log \text{cap}(K_0) - h_0 \log \frac{\text{cap}(K_0)}{\text{cap}(K_1)}, \quad (10.96)$$

which together with (10.93) proves that

$$\text{cap}(K_{h_0}) < \text{cap}(K_0). \quad (10.97)$$

If we set  $\tilde{D} := D_{h_0}$ , then the proposition follows from (10.97) together with (10.94). Hence, it remains only to prove that the two assertions (10.94) and (10.95) hold true.

The two assertions (10.94) and (10.95) have already been proved as assertion (i) and (ii) in Theorem 13, but under partly different assumptions. The difference is the following: In Theorem 13, it has been assumed that both domains  $D_0$  and  $D_1$  belong to  $\mathcal{D}(f, \infty)$ , while in the present situation, we do not know whether  $D_1 \in \mathcal{D}(f, \infty)$ . Instead, we now have the assumptions (i) - (iv), of which (ii) and (iii) are the two most important ones.

In the a step we prove that (10.94) holds true. As in the proof of Lemma 13, where an analogous result has been proved for Proposition 5, the main tool will again be Proposition 5.

It follows from assumption (iii) and the assumption that  $D_0 \in \mathcal{D}(f, \infty)$  together with Proposition 5 that for every Jordan curve  $\gamma \in \Gamma_1$ , with  $\Gamma_1$  introduced in Definition 11, and  $\gamma \subset \overline{\mathbb{C}} \setminus E_0$ , we have

$$\gamma \cap K_0 \neq \emptyset \quad \text{and} \quad \gamma \cap K_1 \neq \emptyset. \quad (10.98)$$

From the defining relations (9.139) - (9.145) in Definition 17 for the sets  $K_h$ , we further conclude that besides of (10.98) we also have

$$\gamma \cap K_h \neq \emptyset \quad \text{for all } h \in (0, 1). \quad (10.99)$$

Details of the argumentation are the same as those in the proof of Lemma 11.

From Proposition 5 and (10.99) it is clear that for a proof of (10.94) it remains only to show that assertion (i) in Proposition 5 holds true for  $0 \leq h \leq h_0$ , i.e., we have to show that there exists  $h_0 \in (0, 1)$  such that the function  $f$  has a meromorphic continuation to each point of  $D_h$  for  $0 \leq h \leq h_0$ .

We defined the two sets  $B_{jh}$ ,  $j = 0, 1$ ,  $h \in (0, 1)$ , by

$$B_{0,h} := \{ z \in D_h \cap K_3 \mid (1-h)g_0(z) > hg_1(z) \}, \quad (10.100)$$

$$B_{1,h} := \{ z \in D_h \cap K_3 \mid (1-h)g_0(z) < hg_1(z) \}. \quad (10.101)$$

where  $g_j$  denotes the Green function  $g_{D_j}(\cdot, \infty)$ ,  $j = 0, 1$ , in the same way as in Definition 17. In the same way as in the proof of Lemma 12, it follows from (9.142), (9.143), (9.144), and (9.145) that the two sets  $D_3 \cap B_{jn}$ ,  $j = 0, 1$ , are domains, and we have

$$B_{jh} \subset D_j \quad \text{for } j = 0, 1, h \in (0, 1). \quad (10.102)$$

Since it has been assumed that  $D_0 \in \mathcal{D}(f, \infty)$ , the function  $f$  processes a meromorphic continuation into the domain  $D_0$ , which we denote by  $f_0$ . From (10.102) it follows that the function  $f_0$  is defined throughout  $D_3 \cup B_{0,h}$ .

An analogous argumentation is unfortunately not possible for the domain  $D_1$ . Here, we have to follow a different path of argumentation. From the assumed properties of the set  $E_0 \subset \overset{\circ}{E}$  it follows that there exists a domain  $D_2$  with the property that

$$D_2 \supset D_3, \quad K_0 \setminus E_0 \subset D_2, \quad (10.103)$$

and the function  $f$  can meromorphically be continued to the domain  $D_2$ . We denote this continuation by  $f_2$ .

In the next step we show that there exists  $h_0 \in (0, 1)$  such that

$$K_h \setminus \text{Int}(E) \subset D_2 \quad \text{for all } 0 \leq h \leq h_0. \quad (10.104)$$

Let  $\tilde{U}_0 \subset D_2$  be an open set consisting only of simply connected components and assume that  $K_0 \setminus U \subset \tilde{U}_0$ , where  $U$  is the open set introduced in the formulation of the proposition. There exists an open set  $U_0 \subset \mathbb{C}$  consisting only of simply connected components such that

$$K_0 \subset U_0, \quad U_0 \cap D_0 \subset \tilde{U}_0, \quad \text{and } U_0 \cap \partial U \subset \tilde{U}_0. \quad (10.105)$$

With exactly the same arguments as those applied in the proof of assertion (iii) of Theorem 13 we then show that there exists  $h_0 \in (0, 1)$  such that

$$K_h \subset U_0 \quad \text{for all } 0 \leq h \leq h_0. \quad (10.106)$$

Notice that in the proof of assertion (iii) of Theorem 13 only topological assumptions about the two sets  $K_0$ ,  $K_1$ , and their complementary domains  $D_0$  and  $D_1$  have been used. The inclusion (10.104) follows directly from (10.106).

Since  $U_0$  consists only of simply connected components, it follows from (10.104), (10.106), and the definition of the set  $B_{1,h}$  in (10.101) that

$$B_{1,h} \subset D_2 \quad \text{for } 0 \leq h \leq h_0, \quad (10.107)$$

and consequently the functions  $f_2$  is defined throughout the domain  $D_3 \cup B_{1,h}$  for all  $0 \leq h \leq h_0$ .

Since  $D_h = D_3 \cup B_{0,h} \cup B_{0,h}$ , we have shown that the function  $f$  possesses a meromorphic continuation to each point  $z \in D_h$  for  $0 \leq h \leq h_0$ . Hence, assumption (i) of Proposition 5 holds true for each  $0 \leq h \leq h_0$ , and (10.94) then follows from Proposition 5.

After the verification of (10.94), it remains only to prove that the inequality (10.95) holds true. Here, we copy the corresponding proof of (9.148) from the proof of Theorem 13 word for word. The condition (9.147) in Theorem 13 follows from the two assumptions (i) and (iv) in Proposition 12. A detailed argumentation for this last conclusion has been given after (10.52). With the proof of (10.95), the whole proof of Proposition 12 is completed.  $\square$

**PROOF OF PROPOSITION 13.** It is rather immediate that the  $S$ -property of  $K$  on  $K \setminus E$  implies that the function  $q$  is analytic in  $\overline{\mathbb{C}} \setminus E$ .

From the definition of  $q$  in (10.44) it follows that level-lines of the Green function  $g_D(\cdot, \infty)$  are trajectories of the quadratic differential  $q(z)dz^2$  (for a definition of trajectories see (5.2) in Section 5.2). The arcs  $J_j$ ,  $j \in I$ , in  $K \setminus E$  are critical trajectories of  $q(z)dz^2$ , and, of course, they are also level-lines of  $g_D(\cdot, \infty)$  corresponding to the value 0. From the local structure of trajectories, it follows that at each bifurcation point  $z \in \overline{\mathbb{C}} \setminus E$  of trajectories, we have a zero of order  $i(z) - 2$ , where  $i(z)$  is the bifurcation index of Definition 6 (cf. [10], Chapter 8.2, or [40]).

That the only zeros of  $q$  in  $\overline{\mathbb{C}} \setminus (E \cup E_1)$  are critical points of the Green function  $g_D(\cdot, \infty)$  in the sense of Definition 7 is an immediate consequence of (10.44), and the same is also true for the double zero of  $q$  at infinity.

We now come to the proof of the inequality (10.45). From (10.44) and we deduce that

$$\begin{aligned} |q(z)| &= 4 \frac{\partial}{\partial z} g_D(z, \infty) \overline{\frac{\partial}{\partial z} g_D(z, \infty)} \\ &= \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) g_D(z, \infty) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) g_D(z, \infty) \\ &= \left( \frac{\partial}{\partial x} g_D(z, \infty) \right)^2 + \left( \frac{\partial}{\partial y} g_D(z, \infty) \right)^2 \end{aligned} \quad (10.108)$$

for  $z \in \overline{\mathbb{C}} \setminus E$ , and consequently we have the estimate

$$\begin{aligned} |q(z)| &= \frac{1}{\pi d^2} \left| \iint_{\Delta(z,d)} q(\zeta) dm_\zeta \right| \leq \frac{1}{\pi d^2} \iint_{\Delta(z,d)} |q(\zeta)| dm_\zeta \\ &= \frac{2}{d^2} D_{\Delta(z,d)}(g_D(\cdot, \infty)) \end{aligned} \quad (10.109)$$

for every  $z \in \mathbb{C} \setminus E$  with  $0 < d < \text{dist}(z, E)$ ,  $\Delta(z, d) := \{ \zeta \mid |\zeta - z| \leq d \}$ ,  $dm_\zeta$  the area element at the point  $\zeta \in \mathbb{C}$ , and  $D_{\dots}(\cdot)$  denotes the Dirichlet integral introduced in (11.56) in Subsection 11.3, further below.

Let now  $r > 0$  be such that  $K \subset \{ |z| \leq r \}$ , and let further  $z \in \{ |z| \leq r \} \setminus E$ . Then we have  $\text{dist}(z, E) < 2r$ , and consequently  $\Delta(z, d) \subset \{ |z| \leq 3r \} \setminus E$  for all  $0 < d < \text{dist}(z, E)$ . From (10.109) and identity (11.60) in Lemma 37 in Subsection

11.3, further below, we deduce that

$$\begin{aligned} |q(z)| &\leq \frac{2}{d^2} D_{\{|z| \leq 3r\} \setminus E}(g_D(\cdot, \infty)) = \frac{2}{d^2} D_{\{|z| \leq 3r\} \setminus K}(g_D(\cdot, \infty)) \\ &= \frac{2}{d^2} \left( \log(3r) + \log \frac{1}{\text{cap}(K)} + O\left(\frac{1}{r}\right) \right) \quad \text{as } r \rightarrow \infty. \end{aligned} \quad (10.110)$$

The equality in the first line of (10.110) is a consequence of the fact that the set  $K \setminus E$  is of planar Lebesgue measure zero, which follows from the assumed  $S$ -property of  $K$  on  $K \setminus E$ . The second equality in (10.110) follows from (11.60) in Lemma 37. Since  $d < \text{dist}(z, E)$  can be chosen arbitrarily, the inequality (10.45) follows directly from (10.110).  $\square$

**PROOF OF LEMMA 19.** Let  $z \in J_j$ ,  $j \in I$ , the an arbitrary point on the Jordan arc  $J_j$ . We have  $g_D(z, \infty) = 0$ , and from the continuity in  $\overline{\mathbb{C}} \setminus E$  of the function  $q$  introduced in (10.44), it follows that the two normal derivatives  $\partial/\partial n_+$  and  $\partial/\partial n_-$  of  $g_D(\cdot, \infty)$  to both sides of the Jordan arc  $J_j$  at  $z$  are equal in modulus.

Since  $g_D(\cdot, \infty) \geq 0$ , and  $g_D(z, \infty) = 0$  for all  $z \in J_j$ , it follows also that the signs of the two normal derivatives are equal. Putting both conclusions together proves equality (10.41) for all  $z \in J_j$ ,  $j \in I$ , and consequently the  $S$ -property on  $K \setminus E$  in the sense of Definition 19 is proved.  $\square$

**10.3. Proofs of Results from Section 4.** The central result of Section 4 is Theorem 4, the Structure Theorem. As a further result, we have Theorem 5, which addresses a special topological property of the minimal set  $K_0(f, \infty)$ , and Theorem 6, which is the analog of Theorem 5 for Problem  $(\mathcal{R}, \infty^{(0)})$ .

**10.3.1. Proof of Theorem 4.** In the proof of Theorem 4 we start from Theorem 15, which covers the special case that the function  $f$  in Problem  $(f, \infty)$  is algebraic, and which therefore implies that the set  $E_0$  of non-polar singularities is finite. The proof of Theorem 4 can then be seen as a lifting of the special result in Theorem 15 to the general situation. In this process the Propositions 11, 12, and 13 from the last subsection will play a decisive role.

**PROOF OF THEOREM 4.** It is immediate that the extremal domain  $D_0(f, \infty) = \overline{\mathbb{C}} \setminus K_0(f, \infty)$  is elementarily maximal in the sense of Definition 8. By  $E_{00} \subset K_0(f, \infty)$  we denote the subset of all  $z \in \partial K_0(f, \infty)$  that satisfy condition (i) in Definition 8, i.e., for each  $z \in E_{00}$ , there exists at least one meromorphic continuation of the function  $f$  out of the domain  $D_0(f, \infty)$  that has a non-polar singularity at  $z$ . By  $E_0$  we then denote the polynomial-convex hull of  $E_{00}$ , i.e.,

$$E_0 := \widehat{E_{00}}. \quad (10.111)$$

(for a definition of the polynomial-convex hull see Definition 22 in Subsection 11.1, further below.) Since  $K_0(f, \infty)$  is polynomial-convex, we have  $E_0 \subset K_0(f, \infty)$ .

In the sequel we assume that  $E_0 \cap K_0(f, \infty) \neq \emptyset$ ; for otherwise Theorem 4 is trivial.

Let  $U_n \subset \mathbb{C}$ ,  $n = 1, 2, \dots$ , be a sequence of open set with simply connected components and smooth boundaries  $\partial U_n$  such that

$$E_0 \subset U_{n+1} \subset \overline{U}_{n+1} \subset U_n \text{ and } E_0 = \bigcap_{n=1}^{\infty} U_n. \quad (10.112)$$

We define  $E_n := \overline{U}_n$  for  $n \in \mathbb{N}$ . It is immediate that all sets  $E_n$  are polynomial-convex, and each of these sets consists only of finitely many components. The minimal set  $K_0 = K_0(f, \infty)$  defines a connectivity relation on the components of each set  $E_n$ ; we say that two components of  $E_n$  are connected in  $K_0$ , if they are connected in  $K_0 \cup E_n$ . Let  $E_{jn}$ ,  $j = 1, \dots, j_n$ , be the components of  $E_n$  with respect to the connectivity in  $K_0$ , i.e.,

$$E_n = E_{1n} \cup \dots \cup E_{j_n, n}, \quad (10.113)$$

and each  $E_{jn}$  is connected in  $K_0$ .

In each component  $E_{jn}$ ,  $j = 1, \dots, j_n$ , we select points  $z_{jl}$  in the following manner: For a given  $n \in \mathbb{N}$  and any  $m \in \mathbb{N}$  we choose  $j_n$  families of points

$$Z_{jnm} = \{z_{jl}\}_{l=1}^{m_j}, \quad j = 1, \dots, j_n, \text{ with } Z_{nm} := \bigcup_{j=1}^{j_n} Z_{jnm}, \quad m = \sum_{j=1}^{j_n} m_j, \quad (10.114)$$

such that

$$Z_{nm} \subset Z_{n, m+1} \quad \text{and} \quad \bigcap_{M=1}^{\infty} \overline{\bigcup_{m \geq M} Z_{nm}} = \partial E_n, \quad (10.115)$$

i.e., the sets  $Z_{jnm}$ ,  $j = 1, \dots, j_n$ , are asymptotically ( $m \rightarrow \infty$ ) dense in each  $\partial E_{jn}$ ,  $j = 1, \dots, j_n$ . Associated with each point set  $Z_{nm}$ , we define the algebraic function

$$f_{nm}(z) := \sum_{j=1}^{j_n} \left[ \prod_{l=1}^{m_j} \left(1 - \frac{z_{jl}}{z}\right) \right]^{1/m_j} \quad \text{for } n, m = 1, 2, \dots \quad (10.116)$$

Let now  $K_{nm} := K_0(f_{nm}, \infty) \subset \mathbb{C}$  be the minimal set for Problem  $(f_{nm}, \infty)$ ,  $D_{nm} := \overline{\mathbb{C}} \setminus K_{nm}$  the corresponding extremal domain  $D_0(f_{nm}, \infty)$ ,  $g_{nm}$  the Green function  $g_{D_{nm}}(\cdot, \infty)$  in  $D_{nm}$ , and let  $q_{nm}$  be the function

$$q_{nm}(z) = \left( 2 \frac{\partial}{\partial z} g_{nm}(z, \infty) \right)^2 \quad (10.117)$$

that is defined analogously to (10.44), and this definition appears also already earlier in (5.3). Since  $f_{nm}$  is an algebraic function, we know from Theorem 15 that  $q_{nm}$  is a rational function. With Lemma 19, it follows from (10.117) that  $K_{nm}$  possesses the  $S$ -property on  $K_{nm} \setminus Z_{nm}$  in the sense of Definition 19.

For  $m \in \mathbb{N}$  sufficiently large, each component  $E_{jn}$  in (10.113) contains elements of the set  $Z_{nm}$ . The definition of the function  $f_{nm}$  together with the definition of the components  $E_{jn}$  in (10.113) then imply that components of  $E_n$  are connected in  $K_{nm}$  for  $m$  sufficiently large if they are also connected in  $K_0 = K_0(f, \infty)$ , i.e., the connectivity defined by  $K_{nm}$  is coarser than that defined by  $K_0$ . Further, it follows from the minimality of  $\text{cap}(K_{nm}) = \text{cap}(K_0(f_{nm}, \infty))$  that the connectivity defined by  $K_{nm}$  is also only minimally coarser in the sense of Definition 20 than that defined by  $K_0$ . Notice that the connectivity relation defined by  $K_0$  on  $E_n$  stands in the background of the definition of the function  $f_{nm}$ .



We will now investigate in a first step what happens with the functions  $f_{nm}$ ,  $g_{nm}$ , and  $q_{nm}$  if  $m \rightarrow \infty$ . In a second step we then will also consider the limits for  $n \rightarrow \infty$ .

From Proposition 13 we deduce the upper estimates

$$|q_{nm}(z)| \leq \frac{2}{\text{dist}(z, E_n)^2} \left( \log(3r) + \log \frac{1}{\text{cap}(K_{nm})} \right) \quad (10.118)$$

for the functions  $q_{nm}$ ; they hold for all  $z \in \{|z| \leq r\} \setminus E_n$ , all  $m \geq m_0$ , and  $r > 0$  sufficiently large. Notice that the assumption  $E_0 \cap K_0(f, \infty) \neq \emptyset$  at the beginning of this proof implies that  $E_0$  contains at least two different components that are connected in  $K_0(f, \infty)$ , which then further implies that  $\text{cap}(K_{nm}) \geq c_0 > 0$  for all  $m \geq m_0$ .

From (10.118) together with Montel's Theorem and the fact that all  $q_{nm}$  are analytic outside of  $E_n$  (cf. Theorem 15, part (b)), we deduce that there exists an infinite sequence  $N_n \subset \mathbb{N}$  such that

$$\lim_{m \rightarrow \infty, m \in N_n} q_{nm}(z) =: q_n(z) \quad (10.119)$$

holds locally uniformly in  $\overline{\mathbb{C}} \setminus E_n$ .

For the Green functions  $g_{nm}$  and the sets  $K_{nm}$  we now prove the existence of limits that correspond to limit (10.119). Using the same techniques as applied in the proof of Lemma 5 after (9.21) and combining this with (10.119) and (10.117), we deduce that the limit

$$\lim_{m \rightarrow \infty, m \in N_n} g_{nm}(z) =: g_n(z) \quad (10.120)$$

exists locally uniformly in  $\mathbb{C} \setminus E_n$ . From the two relations in (10.115) together with the fact that all points in each set  $Z_{jnm} \subset E_{jn}$ ,  $j = 1, \dots, j_n$ , are connected by a subcontinuum in  $K_{nm}$  and that these continua contain only regular boundary points, it follows that

$$g_n(z) = 0 \quad \text{for all } z \in E_n. \quad (10.121)$$

With the two definitions

$$K_n := \bigcap_{M=1}^{\infty} \bigcup_{m \geq M, m \in N_n} K_{nm} \quad \text{and} \quad D_n := \overline{\mathbb{C}} \setminus K_n, \quad (10.122)$$

it further follows from Lemma 42 from Subsection 11.4, further below, that the function  $g_n$  introduced in (10.120) is the Green function of the domain  $D_n$ , i.e., we have

$$g_n = g_{D_n}(\cdot, \infty). \quad (10.123)$$

The two limits (10.119) and (10.120) together imply that analogously to (10.117) we also have the relation

$$q_n(z) = \left( 2 \frac{\partial}{\partial z} g_n(z, \infty) \right)^2 \quad \text{for } z \in \overline{\mathbb{C}} \setminus E_n, \quad (10.124)$$

from which we deduce again with Lemma 19 that the set  $K_n$  possesses the  $S$ -property on  $K_n \setminus E_n$  in the sense of Definition 19.

Like the sets  $K_{nm}$ , so also the set  $K_n$  possesses the property that the connectivity relation defined on the components of  $E_n$  by  $K_n$  is minimally coarser in the sense of Definition 20 than that defined by  $K_0 = K_0(f, \infty)$ . Indeed, both aspects of the property "minimally coarser" carry over from  $K_{nm}$  to  $K_n$ . With

exactly the same techniques as those used in the proof of Lemma 7, we show that all connectivities defined by the sets  $K_{nm}$  lead to identical connectivities defined by the set  $K_n$ . Consequently, also the new connectivity relation is coarser than that defined by the set  $K_0$ . On the other hand, as a consequence of convergence  $\lim_m \text{cap}(K_{nm}) = \text{cap}(K_n)$ , which follows from (10.120), it then further follows that the connectivity defined on  $E_n$  by  $K_n$  is also only minimally coarser in the sense of Definition 20 than that defined by  $K_0$ .

In a second step we investigate limits for  $n \rightarrow \infty$ , where we can largely apply the same techniques as those just used after (10.118) for the investigation of limits with  $m \rightarrow \infty$ , only that now the boundary behavior of the Green function can be complicated by irregular points on  $\partial E_0$ . Notice that the sets  $E_n$ , have been constructed with nice boundaries. We overcome these new difficulties by using Lemma 41 from Subsection 11.4, further below.

From the definition of the sets  $E_n$  after (10.112) we know that

$$E_0 = \bigcap_n E_n. \quad (10.125)$$

Analogously to (10.118), we deduce from Proposition 13 that

$$|q_n(z)| \leq \frac{2}{\text{dist}(z, E_n)^2} \left( \log(3r) + \log \frac{1}{\text{cap}(K_n)} \right) \quad (10.126)$$

for all  $z \in \{|z| \leq r\} \setminus E_n$ ,  $n \in \mathbb{N}$ , and  $r > 0$  sufficiently large, for the functions  $q_n$  from (10.119), which satisfy (10.124). With the same argumentation as used after (10.118), we conclude that  $\text{cap}(K_n) \geq c_0 > 0$  for all  $n \in \mathbb{N}$ , which shows that the right-hand side of (10.126) can be made independent of  $n$ . From (10.126) it then follows as before in (10.119) that there exists an infinite sequence  $N \subset \mathbb{N}$  such that the limit

$$\lim_{n \rightarrow \infty, n \in N} q_n(z) =: \tilde{q}(z) \quad (10.127)$$

exists locally uniformly for  $z \in \overline{\mathbb{C}} \setminus E_0$ , and the function  $\tilde{q}$  is analytic in  $\overline{\mathbb{C}} \setminus E_0$ . With the same argumentation as used for the deduction of (10.120), we now deduce that the limit

$$\lim_{n \rightarrow \infty, n \in N} g_n(z) =: \tilde{g}(z) \quad (10.128)$$

exists locally uniformly for  $z \in \mathbb{C} \setminus E_0$ .

From (10.128) together with Lemma 41 from Subsection 11.4 and (10.125), we deduce that

$$\tilde{g}(z) = 0 \quad \text{for quasi every } z \in E_0. \quad (10.129)$$

Using now the definitions

$$\tilde{K} := \bigcap_{m=1}^{\infty} \bigcup_{n \geq m, m \in N} K_n \quad \text{and} \quad \tilde{D} := \overline{\mathbb{C}} \setminus \tilde{K}, \quad (10.130)$$

it follows in the same way as after (10.122) with the help of Lemma 42 from Subsection 11.4 that the function  $\tilde{g}$  introduced in (10.128) is the Green functions of the domain  $\tilde{D}$ , i.e., we have

$$\tilde{g} = g_{\tilde{D}}(\cdot, \infty). \quad (10.131)$$

Notice that the domain  $\tilde{D}$  may not be regular with respect to Dirichlet problems, there may exist irregular points (cf. Definition 24 in Subsection 11.3, further below)

on  $\partial\tilde{D} \cap E_0$ , and consequently, the equality in (10.129) holds only quasi everywhere on  $E_0$ .

The two limits (10.127) and (10.128) together with relation (10.124) imply that

$$\tilde{q}(z) = \left( 2 \frac{\partial}{\partial z} g_{\tilde{D}}(z, \infty) \right)^2 \quad \text{for } z \in \overline{\mathbb{C}} \setminus E_0. \quad (10.132)$$

With Lemma 19, we deduce from (10.132) that the set  $\tilde{K}$  possesses the  $S$ -property on  $\tilde{K} \setminus E_0$  in the sense of Definition 19. In the same way, as it has been shown before in the cases of the sets  $K_{nm}$  and the set  $K_n$ , we also now show that the minimal coarseness of a connectivity relation in the sense of Definition 20 can be carried over from the connectivity relations defined by the sets  $K_n$  to the relation defined by the set  $\tilde{K}$ , i.e., we deduce that the connectivity defined by  $K_n$  on  $E_0$  is minimally coarser in the sense of Definition 20 than that defined by  $K_0(f, \infty)$ .

Since we have assumed at the beginning of the present proof that  $E_0 \cap K_0(f, \infty) \neq \emptyset$ , it follows that some components of  $E_0$  are connected in  $K_0(f, \infty)$ , and consequently, those components are also connected in  $\tilde{K}$ , which implies that

$$\text{cap}(\tilde{K}) > 0. \quad (10.133)$$

Let us summarize what we have proved so far. Starting from the algebraic functions  $f_{nm}$  introduced in (10.116) and using the results proved in Theorem 15 for algebraic functions, we have shown that there exists a compact set  $\tilde{K} \subset \mathbb{C}$  of positive capacity with the following properties:

- (a)  $\tilde{K}$  possesses the  $S$ -property in the sense of Definition 19 on  $\tilde{K} \setminus E_0$ .
- (b) The connectivity relation defined on  $E_0$  by  $\tilde{K}$  is minimally coarser in the sense of Definition 20 than that defined by  $K_0(f, \infty)$ .
- (c) The set  $E_0 \subset \tilde{K}$  is compact and polynomial-convex, and its boundary  $\partial E_0$  contains all non-polar singularities of the function  $f$  that can be reached by meromorphic continuations of  $f$  from within the extremal domain  $D_0(f, \infty)$ .

In the concluding part of the proof, Proposition 12 will play a crucial role. If in Proposition 12 we take  $K_0(f, \infty)$  as  $K$  and  $\tilde{K}$  from (10.130) as  $K_1$ , then it is immediate from (10.133) and the assertions (a), (b), and (c) that all assumptions of Proposition 12 are satisfied, except the assumption (iv).

Since the set  $K_0(f, \infty)$  is of minimal capacity in the sense of (2.1) in Definition 2, it follows that conclusion (10.43) of Proposition 12 cannot be true. Consequently, assumption (iv) of Proposition 12 has to be false, which proves with our choice of  $K$  and  $K_1$  that we have

$$K_0(f, \infty) = \tilde{K}. \quad (10.134)$$

Indeed, from the negation of assumption (iv) in Proposition 12 we conclude that  $\tilde{K} \subset K_0(f, \infty)$ . Identity (10.134) then follows from a combination of the fact that the connectivity relation defined by  $\tilde{K}$  on  $E_0$  is coarser than that defined by  $K_0(f, \infty)$  (cf. assertion (b)), and the fact that the extremal domain  $D_0(f, \infty)$  is elementarily maximal in the sense of Definition 8 (cf. Proposition 4).

With identity (10.134), all properties proved for the set  $\tilde{K}$  are now also valid for the minimal set  $K_0(f, \infty)$ . Hence, we have proved the following four assertions:

- ( $\alpha$ ) The set  $K_0(f, \infty) \setminus E_0$  consists of critical trajectories of the quadratic differential  $\tilde{q}(z)dz^2$  with  $\tilde{q}$  defined in (10.132). In (10.132),  $\tilde{D}$  is equal to the extremal domain  $D_0(f, \infty)$  because of (10.134).

Indeed, assertion ( $\alpha$ ) follows immediately from (10.132) and the definition of trajectories of quadratic differentials in (5.2) in Subsection 5.2.

- ( $\beta$ ) At each  $z \in \partial E_0$  at least one meromorphic continuation of the function  $f$  out of the extremal domain  $D_0(f, \infty)$  hits a non-polar singularity.

Indeed, assertion ( $\beta$ ) is an immediate consequence of the definition of the set  $E_0$  in and before (10.111).

- ( $\gamma$ ) Let  $E_1$  be set of all zeros of the function  $\tilde{q}$  from (10.132) on  $K_0(f, \infty) \setminus E_0$ . The set  $E_1$  is discrete in  $K_0(f, \infty) \setminus E_0$  since the function  $\tilde{q}$  is analytic in  $\mathbb{C} \setminus E_0$ . The set  $K_0(f, \infty) \setminus (E_0 \cup E_1)$  consists of open, analytic Jordan arcs, which are trajectories of the quadratic differential  $\tilde{q}(z)dz^2$ .

Indeed, the first part of assertion ( $\gamma$ ) follows directly from (10.132). Since  $\tilde{q}(z_0) \neq 0$  for any  $z_0 \in K_0(f, \infty) \setminus (E_0 \cup E_1)$ , we also have  $\frac{\partial}{\partial z} g_{D_0(f, \infty)}(z_0, \infty) \neq 0$ , and consequently, the equation  $g_{D_0(f, \infty)}(z, \infty) = 0$  defines an analytic Jordan arc in a neighborhood of any point  $z_0 \in K_0(f, \infty) \setminus (E_0 \cup E_1)$ , which proves the second part of assertion ( $\gamma$ ).

- ( $\delta$ ) Let  $o(z)$ ,  $z \in E_1$ , be the order of the zero of  $\tilde{q}$  at  $z$ , then the point  $z$  is endpoint of  $o(z) + 2$  different analytic Jordan arcs in  $K_0(f, \infty) \setminus (E_0 \cup E_1)$ .

Indeed, assertion ( $\delta$ ) is an immediate consequence of the typical local structure of trajectories of quadratic differentials in a neighborhood of a zero of the differential (cf. [10], Chapter 8.2, or [40]).

- ( $\varepsilon$ ) The function  $f$  has meromorphic continuations to each point of  $z \in K_0(f, \infty) \setminus E_0$  from all sides out of the extremal domain  $D_0(f, \infty)$ . These continuations lead to exactly two different function elements at each point  $z \in K_0(f, \infty) \setminus (E_0 \cup E_1)$ , and it leads to more than two different function elements at each point  $z \in E_1$ .

Indeed, the first part of assertion ( $\varepsilon$ ) is a consequence of the definition of the set  $E_0$  in (10.111). The second part is a consequence of the minimality (2.1) in Definition 2 of the set  $K_0(f, \infty)$ .

With the four assertions ( $\alpha$ ) - ( $\varepsilon$ ) we have proved more than is needed for the proof of Theorem 4. The additional results will be needed in subsequent proofs, most importantly in the proofs of the two Theorems 7 and 8.

Identity (4.1) in Theorem 4 follows directly from the three assertions ( $\alpha$ ), ( $\beta$ ), and ( $\gamma$ ). The description of the set  $E_0$  in assertions (i) of Theorem 4 is practically identical with assertion ( $\beta$ ). The two remaining assertions (ii) and (iii) in Theorem 4 follow from the two assertions ( $\gamma$ ) and ( $\varepsilon$ ). With the last sentences the proof of Theorem 4 is completed.  $\square$

**10.3.2. Proof of Theorem 5.** The catchword in Theorem 5 is local connectedness. Among other things it will be shown in Theorem 5 that the most interesting part  $K_0(f, \infty) \setminus E_0$  of the minimal set  $K_0(f, \infty)$  is locally connected. Local connectedness is an important property in geometric function theory.

PROOF OF THEOREM 5. Since we know that the function  $\tilde{q}$  of (10.132) is analytic in  $\overline{\mathbb{C}} \setminus E_0$ , it follows from  $\overline{\mathbb{C}} \setminus E_0$  that the Green function  $g_{D_0(f, \infty)}(\cdot, \infty)$  is continuous throughout  $\overline{\mathbb{C}} \setminus E_0$ . From Carathéodory's theory about the boundary behavior of Riemann mapping functions (cf. Chapter 9 in [24], and there especially Theorem 9.8) we know that the continuity of the Green function  $g_{D_0(f, \infty)}(\cdot, \infty)$  is equivalent to the local connectedness of the set  $\partial D_0(f, \infty) \setminus E_0$ . From Theorem 4 it follows that the set  $\partial D_0(f, \infty) \setminus E_0$  is equal to  $K_0(f, \infty) \setminus E_0$ , which proves the first half of Theorem 5.

From the local behavior of trajectories of quadratic differentials, as it has been stated in Lemma 43 in Subsection 11.5, further below, together with assertion  $(\gamma)$  at the end of the proof of Theorem 4, we know that the bifurcation points  $z \in E_1$  of  $K_0(f, \infty) \setminus E_0$  are zeros of the analytic function  $\tilde{q}$  from (10.132), and as such they are of finite order. From the connection between the zeros of quadratic differentials and the local structure of its trajectories (see again Lemma 43 in Subsection 11.5), we then conclude that the finiteness of the order of the zeros of  $\tilde{q}$  implies that each  $z \in E_1$  can be the endpoint of only finitely many Jordan arcs  $J_j$ ,  $j \in I$ .  $\square$

10.3.3. *Proof of Theorem 6.* In Theorem 3 of Section 3 it has been shown that the two Problems  $(f, \infty)$  and  $(\mathcal{R}, \infty^{(0)})$  are equivalent if the Riemann surface  $\mathcal{R}$  is the natural domain of definition for the function  $f$ . Because of this equivalence, Theorem 6 is practically a corollary to Theorem 4.

PROOF OF THEOREM 6. In order to prove the characterization of the sets  $E_0$ ,  $E_1$ , and the family of Jordan arcs  $\{J_j\}_{j \in I}$  in (4.2) of Theorem 6, we have only to keep in mind that a non-polar singularity of the function  $f$  at a point  $z \in \overline{\mathbb{C}}$  is either a branch point or a transcendental, essential singularity. In the first case, the point  $z$  is an element of  $Br(\mathcal{R})$ , and in the second case, it is an element of the relative boundary  $\partial \mathcal{R}$  of the Riemann surface  $\mathcal{R}$  over  $\overline{\mathbb{C}}$ . With these observations we immediately verify identity (4.3) in Theorem 6.

The two assertions (ii) and (iii) in Theorem 6 follow then from the observation that meromorphic continuations of the function  $f$  lead to different function elements at a point  $z \in \overline{\mathbb{C}}$  if, and only if, the corresponding points  $\zeta$  on the Riemann surface  $\mathcal{R}$  lie on different sheets of this surface.  $\square$

**10.4. Proofs of Results from Section 5.** In Section 5 the Jordan arcs  $J_j$ ,  $j \in I$ , in the minimal set  $K_0(f, \infty)$  have been characterized with the help of the  $S$ -property and with the help of quadratic differentials. Both concepts are very similar. These results are especially interesting if the function  $f$  has only finitely many non-polar singularities; Proposition 3 and Theorem 9 deal with this situation.

10.4.1. *Proof of Theorem 7.* Most of the work for a proof of Theorem 7 has already been done in the proof of Theorem 4.

PROOF OF THEOREM 7. In assertion  $(\gamma)$  at the end of the proof of Theorem 4, it has been shown that each Jordan arc  $J_j$ ,  $j \in I$ , is a trajectory of the quadratic differential  $\tilde{q}(z)dz^2$  with the function  $\tilde{q}$  defined by (10.132). Because of (10.134), the domain  $\tilde{D}$  in (10.132) is equal to the extremal domain  $D_0(f, \infty)$ . With Lemma 19, we then conclude from (10.132) together with (10.134) that  $K_0(f, \infty)$  possesses the

$S$ -property in this sense of Definition 19 on  $K_0(f, \infty) \setminus E_0$ , which proves identity (5.1) because of (10.41).  $\square$

10.4.2. *Proof of Theorem 8.* In Theorem 8 the Jordan arcs  $J_j$ ,  $j \in I$ , of the minimal set  $K_0(f, \infty)$  are characterized by quadratic differentials.

PROOF OF THEOREM 8. From (10.134) in the proof of Theorem 4 we know that the function  $\tilde{q}$  introduced in (10.132) and the function  $q$  introduced in (5.3) are identical. In assertion ( $\alpha$ ) at the end of the proof of Theorem 4, it has been proved that the arcs  $J_j$ ,  $j \in I$ , are trajectories of the quadratic differential  $q(z)dz^2$ .

Identity (5.4) in Theorem 8 follows from (10.132) for  $z \in \mathbb{C} \setminus E_0$ , and it follows from the logarithmic pole of the Green functions  $g_{D_0(f, \infty)}(\cdot, \infty)$  at infinity. From (10.127), it further follows that the function  $\tilde{q} = q$  is analytic in  $\overline{\mathbb{C}} \setminus E_0$ . Thus, it only remains to prove that the function  $\tilde{q}$  from (10.132) is meromorphic at isolated points of  $E_0$ . Actually, we shall prove slightly more and show that  $\tilde{q}$  has at most of simple pole at an isolated point of  $E_0$ .

If  $z \in E_0$  is simultaneously an isolated point of  $E_0$  and of the minimal set  $K_0(f, \infty)$ , then the Green function  $g_{D_0(f, \infty)}(\cdot, \infty)$  is harmonic in a neighborhood of  $z$ , and it follows from (10.132) that  $\tilde{q}$  is analytic at  $z$ .

Let us now assume that  $z_0 \in E_0$  is an isolated point of  $E_0$ , but not of the minimal set  $K_0(f, \infty)$ . From assertion ( $\alpha$ ) at the end of the proof of Theorem 4, we know that in a neighborhood of  $z_0$  the set  $K_0(f, \infty) \setminus \{z_0\}$  consists only of trajectories of the quadratic differential  $\tilde{q}(z)dz^2$ . From the history of the definition of the set  $\tilde{K}$  before (10.130), it then follows that only a finite number, let say  $k_0 \in \mathbb{N}$ , of these Jordan arcs  $J_j$ ,  $j \in I$ , in  $K_0(f, \infty) \setminus E_0$  have  $z_0$  as their endpoint. From this observation and the local structure of quadratic differentials, which has been reviewed in Lemma 43 in Subsection 11.5, further below, we conclude that near the point  $z_0$  the function  $\tilde{q}$  of (10.132) has the local development

$$\tilde{q}(z) = q_0(z - z_0)^{k_0-2} + \dots, \quad q_0 \neq 0, \quad (10.135)$$

which shows that the function  $\tilde{q} = q$  is meromorphic at each isolated point of  $E_0$ , and poles are at most of order 1.  $\square$

10.4.3. *Proof of Theorem 9.* In Theorem 9 the special case has been considered that the set  $E_0$  of Theorem 4 is finite, which leads to a rational quadratic differential  $q(z)dz^2$  for the determination of the Jordan arcs  $J_j$ ,  $j \in I$ , in  $K_0(f, \infty) \setminus E_0$  in the sense of Theorem 8. Algebraic functions provide typical examples for this situation.

In Proposition 3 it has been stated that with the set  $E_0$  also the two sets  $E_1$  and  $E_2$  in Theorem 4 and in Definition 7, respectively, are finite.

PROOF OF PROPOSITION 3. If  $E_0$  is finite, then all points of  $E_0$  are isolated, and we know from Theorem 8 that the function  $q$  from (5.3) is meromorphic throughout  $\overline{\mathbb{C}}$ , which implies that  $q$  is a rational function. Together with (5.4) we then further know that its numerator degree is by 2 degrees smaller than its denominator degree. Let  $m$  and  $n$  denote the numerator and denominator degrees,

respectively. Since it has been shown in Theorem 8 that  $q$  has a most simple poles, which are all contained in  $E_0$ , it follows that

$$m + 2 = n \leq \text{card}(E_0). \quad (10.136)$$

From the local structure of the trajectories of quadratic differentials, which has been reviewed in Lemma 43 in Subsection 11.5, further below, we know that at each bifurcation point  $z$  of  $K_0(f, \infty) \setminus E_0$ , the function  $q$  has a zero of order

$$\text{ord}(z) = i(z) - 2 \quad (10.137)$$

with  $\text{ord}(z)$  denoting the order of the zero at  $z$  and  $i(z)$  denoting the bifurcation index of Definition 6. Since  $E_1$  is the set of all bifurcation points of  $K_0(f, \infty)$ , it follows from (10.136) and (10.137) that the set  $E_1$  is finite.

From the definition of critical points of a Green function in Definition 7, it follows rather immediately that at such points several level lines of the Green function intersect. Consequently, the function  $q$  has a zero at each critical point  $z$  of the Green function  $g_{D_0(f, \infty)}(\cdot, \infty)$ , but this also follows directly from (5.3), and more precisely, we have

$$\text{ord}(z) = 2j(z), \quad (10.138)$$

where  $\text{ord}(z)$  denotes again the order of the zero at  $z$ , and  $j(z)$  is the degree of the critical point introduced in Definition 7. From (10.136) and (10.138), it then follows that the set  $E_2$  is finite.  $\square$

**PROOF OF THEOREM 9.** Most work for the proof of Theorem 9 has already been done in the proof of Proposition 3. From there we know that  $q$  is a rational function. All zeros and poles of the function  $q$  on the minimal set  $K_0(f, \infty)$  are contained in  $E_0 \cup E_1$ . The poles are at most of order 1, and they appear at a point  $z \in E_0$  if  $z$  is endpoint of only one Jordan arc in  $K_0(f, \infty) \setminus (E_0 \cup E_1)$ . For such points  $z$ , we have the bifurcation index  $i(z) = 1$ . After putting the information from (10.136) and (10.137) together, we get the first product in (5.6).

The second product in (5.6) follows from (10.138) and the observation that in the domain  $D_0(f, \infty)$  the function  $q$  is analytic and has zeros only at the critical points of the Green function  $g_{D_0(f, \infty)}(\cdot, \infty)$  and at  $\infty$ . The order of these zeros is given by (10.138) and (5.4).  $\square$

**10.5. Proofs of Results from Section 7.** The main results of Section 7 are contained in Theorem 11, where a characterization of the extremal domain  $D_0(f, \infty)$  has been given with the help of the  $S$ -property, and in Theorem 12, where several geometric estimates have been formulated with the help of convexity.

**10.5.1. Proof of Theorem 11.** Most of the work for the proof of Theorem 11 has already been done by Proposition 10.

**PROOF OF THEOREM 11.** Let  $D \in \mathcal{D}(f, \infty)$  be an admissible domain that possesses the  $S$ -property in the sense of Definition 9. We assume that  $D$  is different from the extremal domain  $D_0(f, \infty)$ , i.e.,

$$D \neq D_0(f, \infty). \quad (10.139)$$

It is an immediate consequence of Theorem 4 that the extremal domain  $D_0(f, \infty)$  is elementarily maximal in the sense of Definition 8. From Proposition 10 together with assumption (10.139), we then conclude that

$$\text{cap}(\partial D) < \text{cap}(\partial D_0(f, \infty)). \quad (10.140)$$

But the last inequality contradicts the minimality (2.1) in Definition 2, which proves Theorem 11.  $\square$

10.5.2. *Proof of Theorem 12.* All results in Theorem 12 are basically a consequence of the fact that the capacity is a monotonically decreasing functions under orthogonal projections, a result which has been reviewed in Lemma 22 in Subsection 11.1, further below.

PROOF OF THEOREM 12. Let

$$L = L_{z_0, v} := \{ z_0 + vt \mid t \in \mathbb{R} \}, \quad z_0, v \in \mathbb{C}, \quad |v| = 1, \quad (10.141)$$

be an arbitrary line in  $\mathbb{C}$ , and denote by  $H_{\pm} = H_{\pm}^L$  the two complementary half-planes of  $L$ , i.e.,  $\mathbb{C} \setminus L = H_+ \cup H_- = H_+^L \cup H_-^L$ , and

$$H_{\pm}^L := \{ z \in \mathbb{C} \mid \pm \text{Im}(\overline{v}(z - z_0)) > 0 \} \quad (10.142)$$

with  $z_0$  and  $v$  the same parameters as those used in (10.141).

By  $\varphi_L : \mathbb{C} \rightarrow \mathbb{C}$  we denote the orthogonal projection (11.7) on  $L$  out of  $H_+$ . On  $L \cup H_-$ ,  $\varphi_L$  is the identity.

Before we come to the individual proofs of the five assertions of Theorem 12, we assemble and prove several preparatory assertions, in which  $E_0$  is the compact set introduced in Theorem 4.

- (a) Let  $D$  be an admissible domain for Problem  $(f, \infty)$ , i.e.,  $D \in \mathcal{D}(f, \infty)$ , and set  $K := \overline{\mathbb{C}} \setminus D$ . If the line  $L$  is such that  $K \subset L \cup H_-^L$ ,  $K \cap L \neq \emptyset$ , and that the function  $f$  has a meromorphic continuation out of  $H_+^L$  into a neighborhood of each  $z \in K \cap L$ , then for every line  $\tilde{L}$ , which is parallel to  $L$ , and for which  $f$  has a meromorphic continuation throughout  $H_+^{\tilde{L}}$ , the compact set  $\tilde{K} := \varphi_{\tilde{L}}(K)$  and the domain  $\tilde{D} := \overline{\mathbb{C}} \setminus \tilde{K}$  are admissible for Problem  $(f, \infty)$ , i.e., we have

$$\tilde{K} = \varphi_{\tilde{L}}(K) \in \mathcal{K}(f, \infty) \quad \text{and} \quad \tilde{D} = \overline{\mathbb{C}} \setminus \tilde{K} \in \mathcal{D}(f, \infty). \quad (10.143)$$

By  $\varphi_{\tilde{L}}$  we have denoted the orthogonal projection associated with  $\tilde{L}$  in accordance to (11.7).

Indeed, assertion (a) and especially (10.143) follows directly from Proposition 5 together with the specific properties of the orthogonal projection (11.7) and the assumptions made in assertion (a).

- (b) If the situation of assertion (a) is given, and if we have  $H_+^L \subsetneq H_+^{\tilde{L}}$ , then it follows that

$$\text{cap}(\tilde{K}) \leq \text{cap}(K). \quad (10.144)$$

and a strict inequality holds in (10.144) if, and only if,

$$\text{cap}(K \cap H_+^{\tilde{L}}) > 0. \quad (10.145)$$

Indeed, assertion (b) follows directly from Lemma 22. For the necessity of condition (10.145) one also needs Lemma 21.



(c) If the line  $L$  is such that

$$E_0 \subset L \cup H_-^L, \quad (10.146)$$

then we have

$$K_0(f, \infty) \subset L \cup H_-^L. \quad (10.147)$$

Indeed, if (10.146) holds true, but (10.147) is false, then we concluded from Theorem 4 that  $K_0(f, \infty) \cap H_+^L$  contains at least an open piece of one of the Jordan arcs  $J_j$ ,  $j \in I$ , from (4.1) in Theorem 4. With Lemma 20 in Subsection 11.1, further below, we then deduce that

$$\text{cap}(K_0(f, \infty) \cap H_+^L) > 0, \quad (10.148)$$

which together with the two assertions (a), (b), and assumption (10.146) further implies that there exists a line  $\tilde{L}$ , like that in assertion (a), such that

$$\varphi_{\tilde{L}}(K_0(f, \infty)) \in \mathcal{K}(f, \infty), \quad (10.149)$$

and, as in assertion (b), we further have

$$\text{cap}(\varphi_{\tilde{L}}(K_0(f, \infty))) < \text{cap}(K_0(f, \infty)). \quad (10.150)$$

Inequality (10.150) together with (10.149) contradicts the minimality (2.1) of the set  $K_0(f, \infty)$  in Definition 2, which completes the proof of assertion (c).

(d) Let  $\text{Ex}(K) \subset K$  denote the set of extreme points of a compact set  $K \subset \mathbb{C}$ . We have

$$\text{Ex}(K_0(f, \infty)) \subset E_0. \quad (10.151)$$

Indeed, for each  $z \in \text{Ex}(K_0(f, \infty))$  there exists a straight line  $L$  such that  $L \cap K_0(f, \infty) = \{z\}$  and  $K_0(f, \infty) \subset L \cup H_-^L$ . From assertion (a) and condition (iii) in Definition 2, we then conclude that any meromorphic continuation of the function  $f$  out of  $H_+^L$  has a non-polar singularity at  $z$ . From the last conclusion we deduce (10.151), but also the slightly stronger assertion (e), which is formulated next.

(e) Let  $L$  be a straight line with the property that  $K_0(f, \infty) \subset L \cup H_-^L$ . If  $K_0(f, \infty) \cap L \neq \emptyset$ , then we also have  $\text{Ex}(K_0(f, \infty)) \cap L \neq \emptyset$ , and at each point  $z \in \text{Ex}(K_0(f, \infty)) \cap L$  the restriction of the function  $f$  to  $H_+^L$  has a non-polar singularity.

For the proof of the assertions (iv) and (v) in Theorem 12 we need a refinement of assertion (e).

(f) Let  $L$  be a straight line with the property that  $\text{cap}(K_0(f, \infty) \cap H_+^L) = 0$ . Then the function  $f$  is single-valued in  $H_+^L \setminus K_0(f, \infty)$ . If further  $K_0(f, \infty) \cap L \neq \emptyset$ , then we also have  $\text{Ex}(K_0(f, \infty) \setminus H_+^L) \cap L \neq \emptyset$ , and the restriction of the function  $f$  to  $H_+^L \setminus K_0(f, \infty)$  has a non-polar singularity at each point  $z \in \text{Ex}(K_0(f, \infty) \setminus H_+^L) \cap L$ .

Indeed, the first part of assertion (f) follows directly from the fact that  $H_+^L \setminus K_0(f, \infty) \subset D_0(f, \infty)$ .

From the assumption that  $\text{cap}(K_0(f, \infty) \cap H_+^L) = 0$  together with Lemma 20 from Subsection 11.1, further below, we conclude that the set  $K_0(f, \infty) \cap H_+^L$  contains no continuum, it is totally disconnected, and consequently, we have  $K_0(f, \infty) \cap H_+^L \subset E_0$  as a consequence of condition (iii) in Definition 2.

If  $K_0(f, \infty) \cap L \neq \emptyset$ , then it is immediate that we also have  $\text{Ex}(K_0(f, \infty) \setminus H_+^L) \cap L \neq \emptyset$ . With the first part of assertion (f), we then conclude in the same

way as in the proof of assertion (e) that the meromorphic continuation of the function  $f$  out of the domain  $H_+^L \setminus K_0(f, \infty)$  has a non-polar singularity at every  $z \in \text{Ex}(K_0(f, \infty) \setminus H_+^L) \cap L$ .

We now come to the individual proofs of the five assertions in Theorem 12, where we will the assertions (a) - (f).

(ii) Assertion (ii) is an immediate consequence of assertion (c).

(i) We prove assertion (i) indirectly. Let us assume that there exists a convex and compact set  $K \subset \mathbb{C}$  such that the function  $f$  has a single-valued meromorphic continuation throughout  $\overline{\mathbb{C}} \setminus K$  and that further

$$K_0(f, \infty) \setminus K \neq \emptyset. \quad (10.152)$$

From (10.152) and the convexity of  $K$  it follows that also

$$\text{Ex}(K_0(f, \infty)) \setminus K \neq \emptyset. \quad (10.153)$$

With assertion (e) we then conclude that the meromorphic continuation of the function  $f$  out of  $\overline{\mathbb{C}} \setminus (K_0(f, \infty) \cup K) \subset D_0(f, \infty)$  has a non-polar singularity at each  $z \in \text{Ex}(K_0(f, \infty)) \setminus K$ , which contradicts the assumption that the function  $f$  is meromorphic throughout  $\overline{\mathbb{C}} \setminus K$ . Hence, assertion (i) is proved.

(iii) Assertion (iii) can be proved like assertion (i), only that now we have to use assertion (f) instead of assertion (e). We give more details since some of the conclusions will also be used in the proof of assertion (v), further below.

Assertion (iii) will be proved indirectly. We assume that  $K \subset \mathbb{C}$  is a convex and compact set,  $E \subset \mathbb{C} \setminus K$  a set that is relatively compact in  $\mathbb{C} \setminus K$ ,  $\text{cap}(E) = 0$ , the function  $f$  has a meromorphic and single-valued continuation throughout  $\overline{\mathbb{C}} \setminus (K \cup E)$ , and

$$(K_0(f, \infty) \setminus E) \setminus K = K_0(f, \infty) \setminus (K \cup E) \neq \emptyset. \quad (10.154)$$

From (10.154) it follows that also

$$\text{Ex}(K_0(f, \infty) \setminus (K \cup E)) \setminus (K \cup E) \neq \emptyset. \quad (10.155)$$

Since  $f$  is single-valued and meromorphic throughout  $D_0(f, \infty) = \overline{\mathbb{C}} \setminus K_0(f, \infty)$ , it follows from assertion (f) that  $f$  has to have a non-polar singularity at each  $z \in \text{Ex}(K_0(f, \infty) \setminus (K \cup E)) \setminus (K \cup E)$ , which contradicts the assumption that  $f$  is meromorphic throughout  $\overline{\mathbb{C}} \setminus (K \cup E)$ .

(iv) Let  $K_{\min}$  be the intersection of all compact sets  $K \subset \mathbb{C}$  that satisfy the assumptions made in assertion (iii). With the usual tools of planar topology one can show that  $K_{\min}$  can also be represented as a denumerable intersection of such sets  $K$ . Like these sets  $K$ , so the set  $K_{\min}$  is also convex and compact, and further it follows from the assumptions made in assertion (iii) together with Lemma 21 in Subsection 11.1 that

$$\text{cap}(K_0(f, \infty) \setminus K_{\min}) = 0. \quad (10.156)$$

We define  $E_{\min} := K_0(f, \infty) \setminus K_{\min}$ , and with this definition, assertion (iv) is proved.

(v) Notice that in assertion (iii) we can choose  $K = K_{\min}$ . With the same argumentation as used after (10.155), we show that the meromorphic continuation of the function  $f$  out of  $D_0(f, \infty) \setminus K_{\min}$  has a non-polar singularity at each  $z \in \text{Ex}(K_0(f, \infty) \cap K_{\min})$ . Form the definition of  $K_{\min}$  as the intersection of all compact

sets  $K \subset \mathbb{C}$  that satisfy the assumptions made in assertion (iii), it follows that  $K_0(f, \infty) \subset K_{\min}$ . Hence, we have proved that  $\text{Ex}(K_{\min}) \subset E_0$ .

From  $\text{cap}(E_{\min}) = 0$  together with Lemma 20 from Subsection 11.1, it follows that the set  $E_{\min}$  is totally disconnected, and consequently, it follows from the Structure Theorem 4 that  $E_{\min} \subset E_0$ , which completes the proof of assertion (v).  $\square$

## 11. Some Lemmas from Potential Theory and Geometric Function Theory

In the present section we assemble definitions and lemmas concerning basic properties and tools from potential theory and from geometric function theory. These tools have been used at several places in the sections above. It is hoped that by its concentration in a separate section the flow of argumentation at earlier places has not been interrupted by argumentations of a rather different flavor, or by references to the literature together with the often necessary reformulations and adaptations of results. A separate compilation is also more convenient and economic with respect to a unified terminology, which unfortunately is not typical for the whole spectrum of the literature in this area. As general references to potential-theoretic results we have used [27], [26], and sometimes also [15]. Towards the end of the present section results become more specific, and some of them require rather technical proofs, which could not be found in the literature with the required specific orientation.

We start with topics related to the (logarithmic) capacity, continue then with logarithmic potentials, Green functions, some special results related to sequences of compact sets, and at last with some remarks on trajectories of quadratic differentials. In the penultimate subsection, Carathéodory's Theorem about kernel convergence will be an important piece.

**11.1. Notations and Basic Properties of Capacity.** The (logarithmic) capacity  $\text{cap}(\cdot)$  is a set function defined on capacitable subsets of  $\mathbb{C}$ , which include Borel sets (cf. [26], Chapter 5 or [27], Chapter I.1). For a compact set  $K \subset \mathbb{C}$  a definition with a strong intuitive flavor can be based on the principle of minimal energy: Let  $\mu$  be a probability measure in  $\mathbb{C}$ ; its energy is defined as

$$I(\mu) := \int \int \log \frac{1}{|z - v|} d\mu(z) d\mu(v). \quad (11.1)$$

The capacity of the compact set  $K \subset \mathbb{C}$  can then be defined as

$$\text{cap}(K) := \exp \left( - \inf_{\mu} I(\mu) \right), \quad (11.2)$$

where the infimum extends over all probability measures  $\mu$  with  $\text{supp}(\mu) \subset K$ .

A special role is played by sets  $E \subset \mathbb{C}$  of capacity zero, which are also known as polar sets. The property of being a set of capacity zero is invariant under Möbius transforms, and thanks to this property, the notion of 'capacity zero' can be extended to the whole Riemann sphere  $\widehat{\mathbb{C}}$ .

**DEFINITION 21.** *A property is said to hold quasi everywhere (written in short as 'qu.e.') on a set  $S \subset \widehat{\mathbb{C}}$  if it holds for every  $z \in S \setminus E$  with  $E$  a set of (outer) capacity zero (cf. [27], Chapter I.1).*

The capacity is monoton with respect to an ordering by inclusions (cf. [26], Theorem 5.1.2). In our investigations we have needed some upper and lower estimates, which are formulated in the next lemma (cf. [26], Theorems 5.3.2, 5.3.4, and 5.3.5).

LEMMA 20. (i) Let  $m(\cdot)$  denote the planar Lebesgue measure in  $\mathbb{C}$ . For any compact set  $K \subset \mathbb{C}$  we have

$$\sqrt{m(K)/\pi} \leq \text{cap}(K) \leq \text{diam}(K)/2. \quad (11.3)$$

(ii) For a continuum  $V \subset \mathbb{C}$  we have

$$\text{diam}(K)/4 \leq \text{cap}(V) \leq \text{diam}(K)/2. \quad (11.4)$$

As a set function, the capacity is not additive, and does also not possess one of the usual subadditivity properties as a weaker substitute for the failing additivity. However, sets of capacity zero are an exception in this respect (cf. [26], Theorem 5.1.4).

LEMMA 21. Let  $K, E \subset \mathbb{C}$  be capacitable sets, and assume that  $\text{cap}(E) = 0$ . Then we have

$$\text{cap}(K \setminus E) = \text{cap}(K \cup E) = \text{cap}(K). \quad (11.5)$$

The denumerable union of capacitable sets of capacity zero is again a set of capacity zero.

Another property, which has been relevant in our investigations, concerns radial projections  $\varphi_r : \mathbb{C} \rightarrow \mathbb{C}$  onto a given disk  $D := \{ |z| \leq r \}$ ,  $r > 0$ , and orthogonal projections on a line  $L := \{ z_0 + vt \mid z_0, v \in \mathbb{C}, |v| = 1, t \in \mathbb{R} \}$  from one side. Let the radial projection  $\varphi_r$  be defined by

$$z \mapsto \varphi_r(z) := \min(r, |z|) \frac{z}{|z|}, \quad (11.6)$$

and the orthogonal projection  $\varphi_L$  be defined by

$$z \mapsto \varphi_L(z) := \begin{cases} z_0 + \text{Re}((z - z_0)\bar{v})v & \text{if } \text{Im}((z - z_0)\bar{v}) > 0, \\ z & \text{else.} \end{cases} \quad (11.7)$$

LEMMA 22. (cf. [25], Chapter 9.3, formula (11)) For any capacitable set  $K \subset \mathbb{C}$  and radial projection  $\varphi_r$  defined in (11.6), we have

$$\text{cap}(\varphi_r(K)) \leq \text{cap}(K), \quad (11.8)$$

and for the orthogonal projection  $\varphi_L$  defined in (11.7), we also have

$$\text{cap}(\varphi_L(K)) \leq \text{cap}(K). \quad (11.9)$$

We have a strict inequality in (11.8) or (11.9) if  $\text{cap}(K \setminus \varphi_r(K)) > 0$  or  $\text{cap}(K \setminus \varphi_L(K)) > 0$ , respectively.

The capacity of a set depends only on the outer boundary of this set, which will become clear from the next definition and the follow-on lemma.

DEFINITION 22. For a bounded set  $S \subset \mathbb{C}$  the polynomial-convex hull  $\hat{S}$  (also denoted by  $\text{Pc}(S)$ ) is defined as the union of  $\bar{S}$  with all bounded components of  $\mathbb{C} \setminus \bar{S}$ . The set  $\partial \hat{S}$  is call the outer boundary, and  $\Omega_S = \mathbb{C} \setminus \hat{S}$  the outer domain of  $S$ . A compact set  $K \subset \mathbb{C}$  is call polynomial-convex if  $K = \hat{K}$ .

The notion 'polynomial-convex hull' hints to the possibility to define this hull by polynomial inequalities. We have

$$\widehat{S} = \{ z \in \mathbb{C} \mid |p(z)| \leq \|p\|_S \text{ for all } p \in \mathcal{P} \} \quad (11.10)$$

where  $\mathcal{P}$  denotes the set of all polynomials and  $\|\cdot\|_S$  the uniform norm on  $S$ .

LEMMA 23. (cf. [26], Theorem 5.1.2) For all compact sets  $K \subset \mathbb{C}$  we have

$$\text{cap}(K) = \text{cap}(\widehat{K}). \quad (11.11)$$

A special property of polynomial-convex sets is the fact that their complement is always a domain. The next lemma addresses a similar topic, but under different circumstances.

LEMMA 24. Let  $S \subset \mathbb{C}$  be a set of capacity zero and  $D \subset \overline{\mathbb{C}}$  a domain, then  $D \setminus S$  is connected. If in addition  $S$  is assumed to be closed in  $D$ , then  $D \setminus S$  is a domain.

PROOF. The lemma has a certain degree of immediate evidence since sets of capacity zero are totally disconnected. However, a formal proof as to take care of the topological difficulties in one or the other way. We will use the tools provided by Lemma 3 in Subsection 9.1, further above.

The assertion that  $D \setminus S$  is connected will be proved indirectly, and for this purpose we assume that the opposite holds true. Then there exist two disjoint open sets  $O_1, O_2 \subset \overline{\mathbb{C}}$  with  $D \setminus S \subset O_1 \cup O_2$  and  $O_j \cap (D \setminus S) \neq \emptyset$  for  $j = 1, 2$ . The set  $\widetilde{K} := D \setminus (O_1 \cup O_2)$  is closed in  $D$  and we have  $\widetilde{K} \subset S$ .

Let  $z_j \in O_j \cap D$ ,  $j = 1, 2$ , be two points, and  $\gamma_0$  a Jordan arc connecting  $z_1$  with  $z_2$  in  $D$ , and let further  $U \subset D$  be a small, open, simply-connected neighborhood of  $\gamma_0$  with  $\overline{U} \subset D$ . The arc  $\gamma_0$  can be extended to a closed Jordan curve  $\gamma_1$  in  $\mathbb{C}$ , and correspondingly  $U$  can be extended to a ring domain  $R \subset \mathbb{C}$  with  $\gamma_1 \subset R$  separating the two components  $A_1$  and  $A_2$  of  $\overline{\mathbb{C}} \setminus R$ . This extension can be done in such a way that  $R \cap \widetilde{K} = U \cap \widetilde{K}$ .

It is immediate that each Jordan curve  $\gamma \subset R$  that separates  $A_1$  from  $A_2$  has to intersect the compact set  $K := R \cap \widetilde{K}$ , for otherwise the two sets  $O_1 \cap D$  and  $O_2 \cap D$  would be connected.

After these preparations, we apply the tools offered in Lemma 3, which then shows that there exists a continuum  $V \subset K$  which is not reduced to a single point, and consequently we have

$$\text{cap}(S) \geq \text{cap}(\widetilde{K}) \geq \text{cap}(V) > 0, \quad (11.12)$$

which contradicts the assumption that  $\text{cap}(S) = 0$ .

If  $S$  is closed in  $D$ , then  $D \setminus S$  is open, and consequently it is a domain.  $\square$

**11.2. Logarithmic Potentials.** Let  $\mu$  be a (Borel) measure with compact supp  $(\mu) \subset \mathbb{C}$ . The (logarithmic) potential of the measure  $\mu$  is defined as

$$p(\mu; z) := \int \log \frac{1}{|z - x|} d\mu(x). \quad (11.13)$$

It is a superharmonic function in  $\mathbb{C}$ , and it is continuous quasi everywhere in  $\mathbb{C}$  for every measure  $\mu$  (cf. [15], Chapter III, Theorem 3.6). In the fine topology it is even continuous throughout  $\mathbb{C}$ , but in our investigations, the concept of fine

topology has not been used. We shall address subtle questions about continuity only in connection with the Green function further below in Subsection 11.3.

Let  $\{\mu_n\}_{n \in \mathbb{N}}$  be a weakly convergent sequence of measures with limit measure  $\mu_0$ ; this is written as

$$\mu_n \xrightarrow{*} \mu_0 \quad \text{as } n \rightarrow \infty. \quad (11.14)$$

With the convergence (11.14) corresponds a specific asymptotic behavior of the potentials  $p(\mu_n; \cdot)$ ,  $n \in \mathbb{N}$ , (cf. [27], Chapter I.6.9), which is known as the Lower Envelope Theorem.

**THEOREM 16** (Lower Envelope Theorem). *If  $\text{supp}(\mu_n) \subset K$  for all  $n \in \mathbb{N}$  with  $K \subset \mathbb{C}$  compact, then from (11.14) it follows that*

$$\liminf_{n \rightarrow \infty} p(\mu_n; z) \geq p(\mu_0; z) \quad (11.15)$$

for all  $z \in \mathbb{C}$ , and equality holds in (11.15) quasi everywhere in  $\mathbb{C}$ .

On a compact set  $K \subset \mathbb{C}$  of positive capacity, there uniquely exists an equilibrium measure  $\omega_K$  (cf. [26], Chapter 3.3), which is the probability measure on  $K$  that minimizes the energy (11.2). Its potential has a typical behavior on  $K$  (cf. [26], Theorem 3.3.4), we have

$$p(\omega_K; z) \begin{cases} = -\log \text{cap}(K) & \text{for quasi every } z \in \widehat{K} \\ > -\log \text{cap}(K) & \text{for all } z \in \Omega_K, \end{cases} \quad (11.16)$$

where  $\Omega_K = \overline{\mathbb{C}} \setminus \widehat{K}$  is the outer domain and  $\widehat{K}$  its polynomial-convex hull of  $K$ . Both objects have been introduced in Definition 22.

In potential theory a special role is played by measures of finite energy, i.e., measures  $\mu$  with  $I(\mu) < \infty$  and  $I(\cdot)$  defined by (11.1). For instance, we have the following result ([26], Theorem 3.2.3).

**LEMMA 25.** *For any measure  $\mu$  of finite energy and any bounded measurable set  $S \subset \mathbb{C}$  with  $\text{cap}(S) = 0$ , we have  $\mu(S) = 0$ .*

*The equilibrium measure  $\omega_K$  of a compact set  $K \subset \mathbb{C}$  with  $\text{cap}(K) > 0$  is of finite energy.*

In potential theory, a number of basic properties are known as principles; a first one has already been stated in Theorem 16. In our investigations we have also needed the next one.

**THEOREM 17** (Principle of Domination). *Let  $\mu_1$  and  $\mu_2$  be two (positive) measures with compact support in  $\mathbb{C}$ , let  $\mu_1$  be of finite energy, and let  $c \in \mathbb{R}$  be a constant. If the inequality*

$$p(\mu_1; z) \leq p(\mu_2; z) + c \quad (11.17)$$

*holds true for  $\mu_1$ -almost every  $z \in \mathbb{C}$ , or if it holds true for quasi every  $z \in \text{supp}(\mu_1)$ , then inequality (11.17) holds true for all  $z \in \mathbb{C}$ .*

**PROOF.** The theorem has been proved in [27], Theorem II.3.2, under the assumption that (11.17) is satisfied  $\mu_1$ -almost everywhere.

If (11.17) holds true for quasi every  $z \in \text{supp}(\mu_1)$ , then it follows from Lemma 29 and from the assumption that  $\mu_1$  is of finite energy that inequality (11.17) holds true also  $\mu_1$ -almost everywhere.  $\square$

The minimum of two potentials can again be represented by a logarithmic potential.

LEMMA 26. *Let  $\mu_1$  and  $\mu_2$  be two (positive) measures, then there exists a (positive) measure  $\mu_0$  and a constant  $r_0 \in \mathbb{R}$  such that*

$$\min(p(\mu_1; \cdot), p(\mu_2; \cdot)) = r_0 + p(\mu_0; \cdot) \quad (11.18)$$

*with  $\|\mu_0\| = \max(\|\mu_1\|, \|\mu_2\|)$ . If the two measures  $\mu_1$  and  $\mu_2$  are of finite energy, then the same is true for  $\mu_0$ .*

PROOF. It is rather immediate that the minimum of two superharmonic functions is again superharmonic. One has only to check the definition of superharmonicity. The lemma then follows from the Poisson-Jensen Formula ([26], Theorem 4.5.1). The determination of  $\|\mu_0\|$  follows from a consideration of  $\min(p(\mu_1; \cdot), p(\mu_2; \cdot))$  near infinity.  $\square$

A broad variety of manipulations is possible in the class of logarithmic potentials if one allows signed measures  $\sigma$  in (11.13).

A signed measure  $\sigma$  is of finite energy, i.e.,  $I(\sigma) < \infty$ , if and only if each of its two components  $\sigma_+$  and  $\sigma_-$  ( $\sigma = \sigma_+ - \sigma_-$ ,  $\sigma_+, \sigma_- \geq 0$ ) is of finite energy.

In order to keep our notations simple, we speak of logarithmic potentials also if there is an additive constant, as for instance, is the case on the right-hand side of (11.18).

LEMMA 27. *Let the two potentials  $p_j$ ,  $j = 1, 2$ , be given by*

$$p_j = r_j + p(\sigma_j; \cdot), \quad j = 1, 2, \quad (11.19)$$

*with  $r_j \in \mathbb{R}$  and  $\sigma_j$ ,  $j = 1, 2$ , signed measures in  $\mathbb{C}$ . The functions  $p_3 := |p_1|$ ,  $p_4 := \max(p_1, 0)$ ,  $p_5 := \min(p_1, 0)$ ,  $p_6 := \max(p_1, p_2)$ , and  $p_7 := \min(p_1, p_2)$  then have representations of the same form as in (11.19) with modified constants  $r_j \in \mathbb{R}$  and signed measures  $\sigma_j$ ,  $j = 3, \dots, 7$ . If the two measures  $\sigma_1$  and  $\sigma_2$  are of finite energy, then the same is true for the five measures  $\sigma_3, \dots, \sigma_7$ .*

PROOF. For the positive and negative components of the two measures  $\sigma_j$ ,  $j = 1, 2$ , we write  $\sigma_{j+}$  and  $\sigma_{j-}$ , respectively, i.e., we have  $\sigma_j = \sigma_{j+} - \sigma_{j-}$ . We consider the potentials  $p_{j+} = r_j + p(\sigma_{j+}; \cdot)$ ,  $p_{j-} = p(\sigma_{j-}; \cdot)$ ,  $j = 1, 2$ . Since we have

$$|p_1| = p_{1+} + p_{1-} - 2 \min(p_{1+}, p_{1-}), \quad (11.20)$$

representation (11.19) for  $p_3$  follows directly from Lemma 26. The representations for  $p_4, \dots, p_7$  follow then as further consequences since we have  $p_4 = \frac{1}{2}(p_1 + |p_1|)$ ,  $p_5 = \frac{1}{2}(p_1 - |p_1|)$ ,  $p_6 = p_1 + \max(p_2 - p_1, 0)$ , and  $p_7 = p_1 + \min(p_2 - p_1, 0)$ . The conclusion about the finite energy of the measures  $\sigma_3, \dots, \sigma_7$  follows from the corresponding conclusion in Lemma 26.  $\square$

An important tool for the work with logarithmic potentials is the balayage technique (sweeping out of a measure) (cf. [27], Chapter II.4). In case of logarithmic potentials, the balayage out of an unbounded domain requires special attention.

DEFINITION 23. *Let  $\mu$  be a measure in  $\mathbb{C}$ .*

- (i) For a bounded domain  $D \subset \mathbb{C}$  with  $\text{cap}(\partial D) > 0$ , by  $\hat{\mu}$  we denote the balayage measure resulting from sweeping the measure  $\mu$  out of the domain  $D$ ; it has its support on  $\partial D \cup \text{supp}(\mu) \setminus D$ , and it is defined by the relation

$$p(\hat{\mu}; z) = p(\mu; z) \quad (11.21)$$

for every  $z \in \overline{\mathbb{C}} \setminus \overline{D}$  and for quasi every  $z \in \partial D$ . The balayage measure  $\hat{\mu}$  is uniquely determined by (11.21) if we assume in addition to (11.21) that  $\hat{\mu}(\text{Ir}(\partial D)) = 0$ , where  $\text{Ir}(\partial D)$  is the set of critical points of  $\partial D$  that will be introduced in Definition 24 in the next subsection.

- (ii) For an unbounded domain  $D \subset \overline{\mathbb{C}}$  with  $\infty \in D$  and  $\text{cap}(\partial D) > 0$ , the concept of balayage is the same as in (i) only that relation (11.21) now has the modified form

$$p(\hat{\mu}; z) = p(\mu; z) + c_1 \quad (11.22)$$

with a constant  $c_1 > 0$  given by

$$c_1 = \int g_D(x, \infty) d\mu(x) \quad (11.23)$$

where  $g_D$  is the Green function in  $D$ , which will be introduced in the next subsection.

With the help of the balayage technique, we can introduce an additional method for manipulating logarithmic potentials which in some sense complements the methods considered in Lemma 27, and which has also been used further above.

LEMMA 28. Let the two logarithmic potentials  $p_j$ ,  $j = 1, 2$ , be given in the form (11.19) with signed measures  $\sigma_1$  and  $\sigma_2$  that are of finite energy, and let further  $D \subset \overline{\mathbb{C}}$  be a (possibly unbounded) open set with connected complement. We define a new function  $p_0$  in a piecewise manner by

$$p_0(z) := \begin{cases} p_1(z) & \text{for } z \in D, \\ p_2(z) & \text{for } z \in \overline{\mathbb{C}} \setminus D. \end{cases} \quad (11.24)$$

If we have

$$p_1(z) = p_2(z) \quad \text{for quasi every } z \in \partial D, \quad (11.25)$$

then there exists a signed measure  $\sigma_0$  in  $\mathbb{C}$  and a constant  $r_0 \in \mathbb{R}$  such that

$$p_0(z) = r_0 + p(\sigma_0; z) \quad \text{for quasi every } z \in \mathbb{C}. \quad (11.26)$$

The measure  $\sigma_0$  is of finite energy, and in (11.26) we have equality everywhere in  $\overline{\mathbb{C}} \setminus \partial D$ . Further we have

$$\sigma_0|_D = \sigma_1|_D \quad \text{and} \quad \sigma_0|_{\mathbb{C} \setminus \overline{D}} = \sigma_2|_{\mathbb{C} \setminus \overline{D}}. \quad (11.27)$$

PROOF. The function

$$d := p_1 - p_2 = r_1 - r_2 + p(\sigma_1 - \sigma_2; \cdot) \quad (11.28)$$

has the form (11.19) with defining measure  $\sigma_1 - \sigma_2$ . If we apply the balayage technique to the measure  $\sigma_1 - \sigma_2$  and sweep this measure out of the domain  $\overline{\mathbb{C}} \setminus \overline{D}$ , than this leads to a balayage measure in  $\overline{D}$  which we denote by  $\hat{\sigma}_{12}$ . With an appropriately chosen constant  $r_{12} \in \mathbb{R}$ , we have

$$\text{supp}(\hat{\sigma}_{12}) \subset \overline{D}, \quad \hat{\sigma}_{12}|_D = (\sigma_1 - \sigma_2)|_D, \quad (11.29)$$

$$d(z) = r_{12} + p(\hat{\sigma}_{12}; z) \quad \text{for quasi every } z \in \partial D, \quad (11.30)$$



and the inequality in (11.30) holds also for all  $z \in D$ . The two statements in (11.29) and (11.30) are a consequence of part (ii) in Definition 23. We define

$$\widehat{d} := r_{12} + p(\widehat{\sigma}_{12}; \cdot). \quad (11.31)$$

Since logarithmic potentials are continuous quasi everywhere in  $\mathbb{C}$  (cf. [15], Chapter III, Theorem 3.6), it follows from (11.25) and (11.28) that  $d(z) = 0$  for quasi every  $z \in \partial D$ , and hence, we deduce from (11.30) that

$$\widehat{d}(z) = 0 \quad \text{for quasi every } z \in \partial D, \quad (11.32)$$

and because of (11.29) further that

$$\widehat{d}(z) = 0 \quad \text{for all } z \in \overline{\mathbb{C}} \setminus \overline{D}. \quad (11.33)$$

From (11.24), (11.30), (11.31), (11.32), and (11.33), it then follows that

$$p_0(z) = p_2(z) + \widehat{d}(z) = r_2 + r_{12} + p(\sigma_2 + \widehat{\sigma}_{12}; z) \quad (11.34)$$

for all  $z \in \overline{\mathbb{C}} \setminus \partial D$  and for quasi every  $z \in \partial D$ , which proves (11.26) if we set

$$\sigma_0 := \sigma_2 + \widehat{\sigma}_{12}. \quad (11.35)$$

Since  $\text{cap}(\overline{D}) > 0$  and since  $\sigma_1 - \sigma_2$  is of finite energy, it follows from (11.22) and (11.23) that the measure  $\widehat{\sigma}_{12}$  is of finite energy, and consequently the same is true for  $\sigma_2 + \widehat{\sigma}_{12}$ . The identities in (11.27) follow from (11.29).  $\square$

We close the present subsection with some estimates of the logarithmic energy (11.1) associated with signed measures. It is important here that the logarithmic kernel in (11.1) is positive definite for signed measures  $\sigma$  with  $\text{supp}(\sigma) \subset K \subset \mathbb{C}$  if  $K$  is a compact set with  $\text{cap}(K) \leq 1$ . In the next lemma estimates have been put together that are relevant in this connection.

LEMMA 29. (i) Let  $K \subset \mathbb{C}$  be a compact set of positive capacity. For all signed measures  $\sigma$  with  $\text{supp}(\sigma) \subset K$  we have

$$I(\sigma) \geq \sigma(K)^2 \log \frac{1}{\text{cap}(K)}. \quad (11.36)$$

(ii) Let  $\sigma$  be a signed measure in  $\mathbb{C}$  with

$$\sigma(\mathbb{C}) = 0, \quad (11.37)$$

and let either  $\text{supp}(\sigma)$  be a compact set or let  $\sigma$  be a signed measure of finite energy, then we have

$$I(\sigma) \geq 0, \quad (11.38)$$

and equality holds in (11.38) if, and only if,  $\sigma = 0$ .

PROOF. Part (ii) of the lemma has been proved in [15], Theorem 1.6.

In a first step of the proof of part (i), we assume the set  $K$  is regular (cf. Definition 24, below). We set  $a := \sigma(K)$ , and define

$$\sigma_0 := \sigma - a \omega_K \quad (11.39)$$

with  $\omega_K$  the equilibrium measure on  $K$ . Consequently, we have  $\sigma_0(\mathbb{C}) = 0$ . From (11.1) it follows that

$$I(\sigma) = I(\sigma_0) + a^2 I(\omega_K) + 2a I(\sigma_0, \omega_K), \quad (11.40)$$

where

$$I(\sigma_0, \omega_K) = \int \int \log \frac{1}{|z - v|} d\sigma_0(z) d\omega_K(v) \quad (11.41)$$

is the mutual energy of the two measures  $\sigma_0$  and  $\omega_K$ , which in case of the equilibrium distribution  $\omega_K$  can be expressed as

$$I(\sigma_0, \omega_K) = \int [-g_\Omega(z, \infty) - \log \text{cap}(K)] d\sigma_0(z) \quad (11.42)$$

with the help of Lemma 32, below. In (11.42),  $\Omega$  is the outer domain of  $K$ .

From the assumption that  $K$  is regular, it follows that  $g_\Omega(z, \infty) = 0$  for all  $z \in K$  (cf. the properties stated in (11.43), further below). From  $\sigma_0(\mathbb{C}) = 0$  and (11.42), it then follows that  $I(\sigma_0, \omega_K) = 0$ . From part (ii) we know that  $I(\sigma_0) \geq 0$ , which together with (11.40) and (11.2) proves (11.36).

If the compact set  $K \subset \mathbb{C}$  is not regular, then it can be approximated from the outside by open sets (cf. [26], Theorem 5.1.3). This implies that for any  $\varepsilon > 0$  there exists a compact set  $\tilde{K} \subset \mathbb{C}$  with  $\partial\tilde{K}$  consisting of piece-wise analytic arcs,  $K \subset \text{Int}(\tilde{K})$ , and  $\text{cap}(\tilde{K}) \leq \text{cap}(K) + \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, (11.36) holds also true in the non-regular case.  $\square$

**11.3. The Green Function.** By  $g_D(\cdot, w)$  we denote the Green function in a domain  $D \subset \overline{\mathbb{C}}$  with logarithmic singularity at  $w \in D$  (for a definition see [26], Chapter 4.4, or [27], Chapter I.4). Somewhat different from the usual definitions, we assume that the Green function  $g_D(\cdot, w)$  is defined throughout  $\overline{\mathbb{C}}$  and also for domains  $D \subset \overline{\mathbb{C}}$  with  $\text{cap}(\partial D) = 0$ . If the domain  $D$  has a boundary  $\partial D$  of positive capacity, then for  $w \in D$  we have

$$g_D(z, w) \begin{cases} = 0 & \text{for quasi every } z \in \partial D \\ > 0 & \text{for all } z \in D \\ = 0 & \text{for all } z \in \overline{\mathbb{C}} \setminus \overline{D}. \end{cases} \quad (11.43)$$

If  $\text{cap}(\partial D) = 0$  and  $w \in D$ , then we define  $g_D(\cdot, w) \equiv \infty$ .

Irregular points of  $\partial D$  with respect to solutions of Dirichlet problems in the domain  $D \subset \overline{\mathbb{C}}$  have required special attention at several places in our investigations. Irregular points are indeed an interesting topic in potential theory. This type of points can be defined in many different ways; one of the possibilities is based on the behavior of the Green function  $g_D(\cdot, w)$  on  $\partial D$  (cf. [26], Chapter 4.2). We use this approach in the next definition.

**DEFINITION 24.** *A point  $z \in \partial D$  is irregular with respect to Dirichlet problems in the domain  $D$  (or short: it is an irregular point of  $\partial D$ ) if  $g_D(z, w) > 0$  for some  $w \in D$ . The set of all irregular points of  $\partial D$  is denoted by  $Ir(\partial D)$ .*

It follows from the existence of the Riemann mapping function (see also [26] Theorem 4.2.1) that if  $D \subset \overline{\mathbb{C}}$  is a simply connected domain and  $\partial D$  is not reduced to a single point, then  $Ir(\partial D) = \emptyset$ .

Often we have had to deal with the outer domains  $\Omega_K$  of a compact set  $K \subset \mathbb{C}$ ; the irregular points of  $\partial\Omega_K$  are elements of  $K$ . In the next definition we repeat certain aspects of Definition 24, but with a refined and a partially new orientation.

DEFINITION 25. Let  $K \subset \mathbb{C}$  be a polynomial-convex set of positive capacity with outer domain  $\Omega_K$ . By  $Ir(K) \subset K$  we denote the set  $Ir(\partial\Omega_K)$  of critical points. This set is broken down into the two subsets

$$Ir_I(K) := Ir(K) \cap \overline{K \setminus Ir(K)} \quad \text{and} \quad Ir_{II}(K) := Ir(K) \setminus \overline{K \setminus Ir(K)}. \quad (11.44)$$

We further define the set of regular points of  $K$  as  $Rg(K) := K \setminus Ir(K)$ .

If  $\text{cap}(K) = 0$ , then we defined  $Ir_{II}(K) := Ir(K) := K$  and  $Ir_I(K) := Rg(K) := \emptyset$ .

We note that the set  $Rg(K)$  introduced in Definition 25 is more comprehensive than the set  $Rg(K) \cap \partial\Omega_K$  of regular points with respect to solutions of Dirichlet problems in  $\Omega_K$ . An important result in potential theory is Kellog's Theorem, which we state here in a somewhat specialized and at the same time also extended version (cf. [26], Theorem 4.2.5 together with Theorem 4.4.9).

LEMMA 30. For a polynomial-convex set  $K \subset \mathbb{C}$  we have  $\text{cap}(Ir(K)) = 0$ , and the Green function  $g_\Omega(\cdot, w)$  is continuous in  $\mathbb{C} \setminus Ir_I(K)$  for every  $w \in \Omega = \Omega_K$ .

As a consequence of the Lemmas 20, 25, and 30, we have the following results about irregular points and Green functions.

LEMMA 31. Let  $K \subset \mathbb{C}$  be a polynomial-convex set with outer domain  $\Omega = \Omega_K$ .

- (i) The set  $Ir_{II}(K)$  is totally disconnected.
- (ii) We have  $\omega_K(Ir(K)) = 0$  for the equilibrium distribution  $\omega_K$  on  $K$ .
- (iii) The Green function  $g_\Omega(\cdot, \infty)$  is harmonic in  $(\Omega \setminus \{\infty\}) \cup Ir_{II}(K) = \mathbb{C} \setminus \overline{Rg(K)}$ .
- (iv) We have  $\text{cap}(U_z \cap K) > 0$  for every open neighborhood  $U_z \subset \mathbb{C}$  of a point  $z \in \overline{Rg(K)}$ , and  $\text{cap}(U_z \cap K) = 0$  for every open neighborhood  $U_z \subset \mathbb{C} \setminus \overline{Rg(K)}$  of a point  $z \in Ir_{II}(K)$ .

PROOF. The first three assertions are rather immediate. Assertion (iv) follows from [26], Theorem 4.2.3 and Theorem 4.2.4.  $\square$

A connection between logarithmic potentials and Green functions is given by the representation formula in the next lemma (cf. [26], Theorem 4.4.7 together with Theorem 5.2.1).

LEMMA 32. Let  $K \subset \mathbb{C}$  be a compact set with  $\text{cap}(K) > 0$ ,  $\Omega = \Omega_K$  its outer domain, and  $\omega_K$  the equilibrium distribution on  $K$ . Then for the Green function  $g_\Omega(\cdot, \infty)$ , we have the representation

$$g_\Omega(\cdot, \infty) = -p(\omega_K; \cdot) + \log \frac{1}{\text{cap}(K)}, \quad (11.45)$$

and near infinity we have

$$g_\Omega(z, \infty) = \log |z| + \log \frac{1}{\text{cap}(K)} + O\left(\frac{1}{|z|}\right) \quad \text{as } z \rightarrow \infty. \quad (11.46)$$

The next result is related to Lemma 32.

LEMMA 33. Let  $K_1, K_2 \subset \mathbb{C}$  be polynomial-convex sets of positive capacity. Then we have

$$g_{\mathbb{C} \setminus K_1}(\cdot, \infty) \equiv g_{\mathbb{C} \setminus K_2}(\cdot, \infty) \quad (11.47)$$

if, and only if,

$$\text{cap}((K_1 \setminus K_2) \cup (K_2 \setminus K_1)) = 0, \quad (11.48)$$

i.e., if, and only if, the two sets  $K_1$  and  $K_2$  differ only in a set of capacity zero.

PROOF. We assume that identity (11.47) holds true. From (11.47) together with the defining identity (11.43) for the Green function and the definition of irregular points in Definition 24 it follows that

$$(\partial K_1 \setminus \partial K_2) \cup (\partial K_2 \setminus \partial K_1) \subset \text{Ir}(K_1) \cup \text{Ir}(K_2), \quad (11.49)$$

which with Lemma 30 and Lemma 21 implies (11.48).

If, on the other hand, (11.48) holds true, then identity (11.47) is an immediate consequence of the defining identity (11.43) for the Green function and the uniqueness of the Green function.  $\square$

The balayage technique of Definition 23 can be made more concrete with the help of the Green function (cf. [27], Chapter II.4).

LEMMA 34. *Under the assumptions of Definition 23, we have*

$$p(\hat{\mu}; \cdot) = p(\mu; \cdot) - \int g_D(\cdot, x) d\mu(x) \quad (11.50)$$

if the domain  $D$  is bounded, and otherwise we have

$$p(\hat{\mu}; \cdot) = p(\mu; \cdot) - \int [g_D(\cdot, x) - g_D(x, \infty)] d\mu(x). \quad (11.51)$$

Related to Lemma 34 is the Riesz Decomposition Theorem (cf., Theorem 3.1 of [27], Chapter II), and more definite the Poisson-Jensen Formula, which has been used at several places in our investigations.

THEOREM 18. (*Poisson-Jensen Formula*) *Let  $D \subset \overline{\mathbb{C}}$  be a domain with  $\text{cap}(\partial D) > 0$ . We assume that the real-valued function  $u$  is subharmonic on  $\overline{D}$ , not identical  $-\infty$ , and possesses an harmonic majorant in  $\overline{D}$ . Then there exists a nonnegative measure  $\mu$  in  $D$  of finite mass such that*

$$u(z) = - \iint_D g_D(z, v) d\mu(v) + \int_{\partial D} u(v) d\hat{\delta}_z(v) \quad \text{for } z \in D \quad (11.52)$$

with  $\hat{\delta}_z$  denoting the balayage measure on  $\partial D$  resulting from sweeping out the Dirac measure  $\delta_z$  out of  $D$ . ( $\hat{\delta}_z$  is also known as the harmonic measure on  $\partial D$  of the point  $z \in D$ .)

PROOF. The theorem has been proved in [26] as Theorem 4.5.1 under stronger assumptions about the domain  $D$  and the function  $u$ . The more general form of the theorem given here is the consequence of a combination of the two Theorems 4.5.1 and 4.5.4 in [26].  $\square$

Analogously to the energy  $I(\cdot)$  that has been defined in (11.1) with a logarithmic kernel, one can also define an energy formula with a Green kernel, i.e., a formula like (11.1) with a Green function as kernel. The new formula is called Green energy. A systematic investigation of Green energy and Green capacity together with the associated Green potentials can be found in [27], Chapter II.5.

In our investigations, we have needed the property of positive definiteness of the Green kernel. The result is contained in the next lemma, and it can be seen as a completion of the material in Lemma 29.

LEMMA 35. *Let  $D \subset \overline{\mathbb{C}}$  be a domain with  $\text{cap}(\partial D) > 0$ . For a signed measure  $\sigma$  of finite energy we have*

$$\int \int g_D(z, v) d\sigma(z) d\sigma(v) \geq 0 \quad (11.53)$$

*and equality holds in (11.53) if, and only if,  $\sigma|_{D \cup I_T(\partial D)} = 0$ .*

PROOF. The lemma can be proved like the analogous result in [15], Theorem 1.6, together with the tools used in [27] for the proof of Lemma 5.4 in Chapter II.  $\square$

We next come to some results that are connected with the Green formula. Let  $D \subset \overline{\mathbb{C}}$  be a domain with a smooth and non-empty boundary  $\partial D \subset \mathbb{C}$ , and let further  $u$  and  $v$  be two real  $C^2$ -functions in  $D$  with  $L^1$ -integrable second derivatives in  $D$  and  $C^1$  boundary functions with  $L^1$ -integrable first derivatives on  $\partial D$ . Under these assumptions, the Green identity

$$\iint_D u \nabla \nabla v \, dm + \iint_D \nabla u \nabla v \, dm + \int_{\partial D} u \frac{\partial}{\partial n} v \, ds = 0 \quad (11.54)$$

holds true (see Chapter VIII of [11] or [8], Theorem 1.9). In (11.54),  $\nabla \nabla$  denotes the Laplace operator  $\partial^2/\partial x^2 + \partial^2/\partial y^2$ ,  $\nabla$  the Laplace or gradient operator  $(\partial/\partial x, \partial/\partial y)$ ,  $\partial/\partial n$  the inwardly showing normal derivation on  $\partial D$ ,  $dm$  the area element in  $D$ , and  $ds$  the (positively oriented) line element on  $\partial D$ .

If the function  $v$  is harmonic in  $D$ , then obviously the first term in (11.54) vanishes. The second term is known as the Dirichlet integral of  $u$  and  $v$ , and we use the abbreviation

$$D_D(u, v) := \frac{1}{2\pi} \iint_D \nabla u \nabla v \, dm. \quad (11.55)$$

Using the same letter  $D$  in one formula with two different meanings is certainly unlucky, but mix-ups should be avoidable. In comparison to the assumptions made in (11.54), we often relax assumptions for (11.55); thus, for instance, with the help of an exhaustion technique we can admit arbitrary domains  $D \subset \overline{\mathbb{C}}$ . If not explicitly stated otherwise, then we assume that both functions  $u$  and  $v$  in (11.55) have  $L^2$ -integrable first order derivatives almost everywhere in  $D$ .

In the special case that both functions  $u$  and  $v$  are identical, we write

$$D_D(u) := D_D(u, u) = \frac{1}{2\pi} \iint_D (\nabla u)^2 \, dm. \quad (11.56)$$

Notice that in (11.54) the Dirichlet integral  $2\pi D_D(u, v)$  is the only term that is symmetric in both functions  $u$  and  $v$ . The use of this fact leads to interesting special cases of the Green identity. Thus, for instance, one gets Formula (1.1) in Chapter II.1 of [27], which will also be used in the present subsection; it is the basis for the proof of Lemma 36, below, after the next paragraph.

Next, we come to several lemmas that are rather immediate consequences on the Green identity. The first one has been used at several places, where potentials have been defined in a piecewise manner. With respect to a proof of this result, we are in the lucky situation that most of the detailed work has already been done in [27], Chapter II, where similar results have been proved.

LEMMA 36. Let  $D \subset \overline{\mathbb{C}}$  be a domain with  $\text{cap}(\partial D) > 0$ ,  $\gamma$  a  $C^{1+\delta}$ -smooth Jordan arc in  $D$ ,  $\delta > 0$ , and  $u$  a bounded real-valued function that is continuous in  $\overline{D} \setminus \text{Ir}(\partial D)$ , harmonic in  $D \setminus \gamma$ , and which possesses  $L^1$ -integrable normal derivatives to both sides of  $\gamma$ .

If  $u(z) = 0$  for all  $z \in \partial D \setminus \text{Ir}(\partial D)$ , then we have

$$u(z) = -\frac{1}{2\pi} \int_{\gamma} \left( \frac{\partial}{\partial n_-} + \frac{\partial}{\partial n_+} \right) u(v) g_D(z, v) ds_v \quad \text{for } z \in D. \quad (11.57)$$

In (11.57),  $\partial/\partial n_+$  and  $\partial/\partial n_-$  denote the normal derivation to both sides of  $\gamma$ ,  $ds$  is the line element on  $\gamma$ , and  $g_D(\cdot, \cdot)$  the Green function in  $D$ .

PROOF. From Theorem 1.5 in Chapter II of [27] it follows that if we define the measure  $\sigma$  on  $\gamma$  by

$$d\sigma(v) := -\frac{1}{2\pi} \left( \frac{\partial}{\partial n_-} + \frac{\partial}{\partial n_+} \right) u(v) ds_v, \quad (11.58)$$

then the function

$$d(z) := u(z) - \int g_D(z, v) d\sigma(v), \quad z \in D, \quad (11.59)$$

is harmonic in  $D$ . From the assumptions of the lemma and from (11.43) together with Definition 24, we know that  $d(z) = 0$  for all  $z \in \partial D \setminus \text{Ir}(\partial D)$ . Since  $d$  is bounded in  $D$ , it follows from the uniqueness of the Dirichlet problem under the given circumstances (cf. Theorem 3.1 in the Appendix A of [27]) that  $d(z) = 0$  for  $z \in D$ , which proves the lemma.  $\square$

In the next lemmas, properties of Green functions are expressed with the help of Dirichlet integrals.

LEMMA 37. Let  $D \subset \overline{\mathbb{C}}$  be a domain with  $\infty \in D$ ,  $r > 0$ , and  $\text{cap}(\partial D) > 0$ , then we have

$$D_{\{|z| < r\} \cap D}(g_D(\cdot, \infty)) = \log(r) + \log \frac{1}{\text{cap}(\partial D)} + O\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty. \quad (11.60)$$

PROOF. The lemma is a consequence of the Green identity (11.54). In a first step we assume that the domain  $D$  has a sufficiently smooth boundary  $\partial D$ . If  $\partial D$  is  $C^2$  smooth, then the Green function  $g_D(\cdot, \infty)$  has continuous first order derivatives on  $\partial D$  (details can be found, for instance, in the proof of Theorem 4.11 in Chapter II of [27]). For  $r > 0$  sufficiently large, we define

$$D_r := D \setminus \{|z| \geq r\}. \quad (11.61)$$

Since  $g_D(\cdot, \infty)$  is harmonic in  $D_r$ , we deduce from (11.54) and (11.56) that

$$\begin{aligned} D_{\{|z| < r\} \cap D}(g_D(\cdot, \infty)) &= \frac{1}{2\pi} \iint_{D_r} (\nabla g_D(\cdot, \infty))^2 dm \\ &= -\frac{1}{2\pi} \int_{\partial D_r} g_D(\cdot, \infty) \frac{\partial}{\partial n} g_D(\cdot, \infty) ds \\ &= -\frac{1}{2\pi} \int_{\partial D} g_D(\cdot, \infty) \frac{\partial}{\partial n} g_D(\cdot, \infty) ds - \frac{1}{2\pi} \int_{\{|z|=r\}} g_D(\cdot, \infty) \frac{\partial}{\partial n} g_D(\cdot, \infty) ds. \end{aligned} \quad (11.62)$$

From the smoothness of  $\partial D$  it follows that  $\partial D$  is regular, and consequently that  $g_D(z, \infty) = 0$  for all  $z \in \partial D$ , which implies that the first integral in the last line of (11.62) is identical zero.

From Lemma 32 we know that the function

$$\tilde{g} := g_D(\cdot, \infty) - \log |\cdot| \quad (11.63)$$

is harmonic in  $\{|z| > r\}$ , and we have  $\tilde{g}(\infty) = -\log \text{cap}(\partial D)$ . From (11.63) it follows that for the inward showing normal derivative on the circle  $\{|z| = r\}$  we have

$$\frac{\partial}{\partial n} g_D(z, \infty) = \frac{\partial}{\partial n} \tilde{g}(z) + \frac{1}{r} \quad \text{for } |z| = r. \quad (11.64)$$

It is rather immediate that

$$\frac{1}{2\pi} \int_{\{|z|=r\}} \tilde{g}(z) \frac{1}{r} ds_z = \tilde{g}(\infty) = \log \frac{1}{\text{cap}(\partial D)}, \quad (11.65)$$

$$\frac{1}{2\pi} \int_{\{|z|=r\}} \log |z| \frac{1}{r} ds_z = \log(r), \quad (11.66)$$

and since  $\tilde{g}$  is harmonic in  $\{|z| > r\}$ , we further have

$$\int_{\{|z|=r\}} \frac{\partial}{\partial n} \tilde{g}(z) ds_z = 0, \quad (11.67)$$

$$\left\| \frac{\partial}{\partial n} \tilde{g} \right\|_{\{|z|=r\}} = O\left(\frac{1}{r^2}\right) \quad \text{as } r \rightarrow \infty, \quad (11.68)$$

where  $\|\cdot\|_{\{|z|=r\}}$  denotes the sup-norm on  $\{|z| = r\}$ .

Using (11.63) through (11.68), the only remaining integral in the last line of (11.62) can be transformed in the following way:

$$\begin{aligned} & -\frac{1}{2\pi} \int_{|z|=r} g_D(\cdot, \infty) \frac{\partial}{\partial n} g_D(\cdot, \infty) ds = \\ & -\frac{1}{2\pi} \int_{|z|=r} \left( \log |\cdot| \frac{\partial}{\partial n} \tilde{g} + \log |\cdot| \frac{1}{r} + \tilde{g} \frac{\partial}{\partial n} \tilde{g} + \tilde{g} \frac{1}{r} \right) ds = \\ & \log(r) + \log \frac{1}{\text{cap}(\partial D)} + O\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty, \end{aligned} \quad (11.69)$$

which then proves (11.60) under the assumption of a sufficiently smooth boundary  $\partial D$ .

In the general case, the domain  $D$  is exhausted by a nested sequence of domains  $D_n$ ,  $n \in \mathbb{N}$ , with sufficiently smooth boundaries  $\partial D_n$ . We assume that

$$\overline{D_n} \subset D_{n+1} \subset D \quad \text{and} \quad D = \bigcup_n D_n. \quad (11.70)$$

By  $g_n$  we denote the Green function  $g_{D_n}(\cdot, \infty)$ . Because of (11.70) we have

$$g_n(z) \geq g_{n+1}(z) \geq g_D(z, \infty) \quad \text{for } z \in \mathbb{C}. \quad (11.71)$$

From the Harnack principle of monotonic convergence (cf. Theorem 4.10 in Chapter 0 of [27]) and the assumption that  $\text{cap}(\partial D) > 0$ , we then deduce that the sequence  $\{g_n\}$  as well as their first order derivatives  $\nabla g_n$  converge locally uniformly in  $D$ , i.e., we have

$$\lim_{n \rightarrow \infty} \nabla g_n = \nabla g \quad (11.72)$$

locally uniformly in  $D$ . From the identity (11.60) for the domains  $D_n$  together with (11.72), identity (11.60) then follows also in the general case.  $\square$

In (11.43), the Green function  $g_D(\cdot, \cdot)$  has been defined for the whole Riemann sphere  $\overline{\mathbb{C}}$ , and one could therefore consider the extension of the Dirichlet integral in (11.60) from  $\{|z| \leq r\} \cap D$  to the whole disc  $\{|z| \leq r\}$ . Such an extension would indeed be without problems if the planar Lebesgue measure of  $\partial D$  were zero. However,  $\partial D$  may be of positive planar Lebesgue measure.

The combination of the assertion of Lemma 37 for two different domains yields the next corollary, which has been useful for the comparison of the capacities of the complements of two domains.

**COROLLARY 3.** *Let  $D_1, D_2 \subset \overline{\mathbb{C}}$  be two domains with  $\infty \in D_j$  and  $\text{cap}(\partial D_j) > 0$  for  $j = 1, 2$ . Then for  $r > 0$  we have*

$$\begin{aligned} D_{\{|z| < r\} \cap D_1}(g_{D_1}(\cdot, \infty)) - D_{\{|z| < r\} \cap D_2}(g_{D_2}(\cdot, \infty)) = \\ \log \frac{\text{cap}(\partial D_2)}{\text{cap}(\partial D_1)} + O\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty. \end{aligned} \quad (11.73)$$

**LEMMA 38.** *Let the function  $u$  be harmonic and bounded in the domain  $D \subset \overline{\mathbb{C}}$  with  $\infty \in D$  and  $\text{cap}(\partial D) > 0$ . Then the Dirichlet integral  $D_D(u, g_D(\cdot, \infty))$  exists, and we have*

$$\lim_{r \rightarrow \infty} D_{\{|z| < r\} \cap D}(u, g_D(\cdot, \infty)) = D_D(u, g_D(\cdot, \infty)) = 0. \quad (11.74)$$

**PROOF.** Like in the proof of Lemma 37, in a first step, we assume that the domain  $D$  has a  $C^2$  smooth boundary  $\partial D$ . The subdomain  $D_r$  is again defined by (11.61) for  $r > 0$  sufficiently large. Analogously to (11.62), we have the identities

$$\begin{aligned} D_{D_r}(u, g_D(\cdot, \infty)) &= \frac{1}{2\pi} \iint_{D_r} \nabla u \nabla g_D(\cdot, \infty) dm \\ &= -\frac{1}{2\pi} \int_{\partial D_r} u \frac{\partial}{\partial n} g_D(\cdot, \infty) ds \\ &= -\frac{1}{2\pi} \int_{\partial D} u \frac{\partial}{\partial n} g_D(\cdot, \infty) ds - \frac{1}{2\pi} \int_{\{|z|=r\}} u \frac{\partial}{\partial n} g_D(\cdot, \infty) ds \\ &= -u(\infty) + u(\infty) + O\left(\frac{1}{r}\right) = O\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty. \end{aligned} \quad (11.75)$$

Indeed, the second equality is a consequence of the Green identity (11.54). The penultimate equality in (11.75) follows from two observations, which are concerned with the two integrals in the third line of (11.75). In the first integral the normal derivative  $(1/2\pi)\partial g_D(\cdot, \infty)/\partial n$  is the density of the equilibrium distribution on  $\partial D$  (cf. Theorem 4.11 of Chapter II in [27]), and it defines the balayage measure on  $\partial D$  resulting from sweeping  $\delta_\infty$  out of  $D$ , which implies that this first integral is equal to  $-u(\infty)$ .

The second integral in the third line of (11.75) extends over the circle  $\{|z| = r\}$ . Using the definition of the function  $\tilde{g}$  in (11.63) together with (11.64) and (11.68) yields

$$\begin{aligned} -\frac{1}{2\pi} \int_{\{|z|=r\}} u \frac{\partial}{\partial n} g_D(\cdot, \infty) ds &= -\frac{1}{2\pi} \int_{\{|z|=r\}} \left( u(z) \frac{\partial}{\partial n} \tilde{g}(z) + u(z) \frac{1}{r} \right) ds_z \\ &= O\left(\frac{1}{r}\right) + u(\infty) \quad \text{as } r \rightarrow \infty. \end{aligned} \quad (11.76)$$

The last two observation together prove the penultimate equality in (11.75).



We note that the two integrals in the third line of (11.75) have opposite orientations with respect to the two domains  $D$  and  $\{|z| > r\}$ . Like in the conclusions after (11.63), and also in (11.75), we have applied Theorem 3.1 of the Appendix A in [27].

With (11.75) we have proved (11.74) under the assumption of a sufficiently smooth boundary  $\partial D$ . We add that the existence of the integral  $D_{D_r}(u, g_D(\cdot, \infty))$  follows from the Cauchy-Schwartz inequality  $D_{D_r}(u, g_D(\cdot, \infty))^2 \leq D_{D_r}(u) D_{D_r}(g_D(\cdot, \infty))$  together with  $\text{cap}(\partial D) > 0$  and the assumed boundedness of the function  $u$ .

Like in the proof of Lemma 37, for a general domain  $D$  identity (11.74) follows from exhausting the domain  $D$  by a sequence of nested domains  $D_n$ ,  $n \in \mathbb{N}$ , with sufficiently smooth boundaries  $\partial D_n$ .  $\square$

In the next two lemmas we prove rather technical results, which have been used in Subsection 10.2, further above.

**LEMMA 39.** *Let  $D \subset \overline{\mathbb{C}}$  be a domain with  $\infty \in D$  and  $\text{cap}(\partial D) > 0$ . Set  $D_r := \{|z| < r\} \cap D$  with  $r > 0$  sufficiently large so that  $\partial D \subset \{|z| < r\}$ , and let  $u$  be a real-valued function that is harmonic in  $\{|z| > r\}$ , and let further  $\hat{u}_r$  be the solution of the Dirichlet problem in  $D_r$  with boundary function*

$$\hat{u}_r(z) = \begin{cases} 0 & \text{for } z \in \partial D, \\ u(z) & \text{for } |z| = r. \end{cases} \quad (11.77)$$

*Under these assumptions, we have*

$$D_{\{|z| < r\} \cap D}(\hat{u}_r, g_D(\cdot, \infty)) = u(\infty) + O\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty. \quad (11.78)$$

**PROOF.** In a first step, we assume that  $D$  has a  $C^2$  smooth boundary  $\partial D$ . Like in (11.75), we deduce from (11.54) that

$$\begin{aligned} D_{D_r}(\hat{u}_r, g_D(\cdot, \infty)) &= -\frac{1}{2\pi} \int_{\partial D} \hat{u}_r \frac{\partial}{\partial n} g_D(\cdot, \infty) ds - \frac{1}{2\pi} \int_{\{|z|=r\}} \hat{u}_r \frac{\partial}{\partial n} g_D(\cdot, \infty) ds. \end{aligned} \quad (11.79)$$

It follows from the first line in (11.77) that the first integral in the second line of (11.79) is identical zero. For the second integral we deduce with the same arguments as applied in (11.76) and with the use of the last line in (11.77) that

$$-\frac{1}{2\pi} \int_{\{|z|=r\}} \hat{u}_r \frac{\partial}{\partial n} g_D(\cdot, \infty) ds = u(\infty) + O\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty. \quad (11.80)$$

Identity (11.78) follows immediately from (11.79) and (11.80) for a domain  $D$  with a sufficiently smooth boundary  $\partial D$ .

For a general domain, identity (11.78) can again be proved by exhausting the domain  $D$  by a sequence of nested domains  $D_n$ ,  $n \in \mathbb{N}$ , as it has been done in the proof of Lemma 37.  $\square$

**LEMMA 40.** *Let  $D \subset \overline{\mathbb{C}}$  be a domain with  $\text{cap}(\partial D) > 0$ ,  $V \subset \overline{D}$  a compact set,  $\mu$  a positive measure of finite energy with  $\text{supp}(\mu) \subset V$ , and  $u$  a real-valued function defined by*

$$u(z) := h(z) + \int g_D(z, v) d\mu(v) \quad \text{for } z \in D \quad (11.81)$$

with  $h$  a harmonic and bounded function in  $D$ . If we assume that

$$u(z) = 0 \quad \text{for quasi every } z \in V, \quad (11.82)$$

then we have

$$D_{D \setminus V}(u) = D_D(h) + \iint g_D(v, w) d\mu(v) d\mu(w). \quad (11.83)$$

PROOF. We deduce from Lemma 38 that

$$D_D(h, g_D(\cdot, v)) = 0 \quad \text{for } v \in D, \quad (11.84)$$

since with the help of a Moebius transform any  $v \in D$  can be transported to infinity. If we choose  $g_D(\cdot, w)$ ,  $w \in D$ , instead of the function  $h$  in (11.84) and set  $D_r := D \setminus \{|z - w| \leq r\}$ , for  $r > 0$  small, then we deduce from Lemma 38 that

$$D_{D_r}(g_D(\cdot, v), g_D(\cdot, w)) = 0. \quad (11.85)$$

With the same argumentation as used after (11.62) and later also in the proof of Lemma 38 after (11.75), we show that

$$\lim_{r \rightarrow \infty} D_{\{|z - w| < r\}}(g_D(\cdot, v), g_D(\cdot, w)) = g_D(w, v). \quad (11.86)$$

Putting (11.85) and (11.86) together proves that

$$D_D(g_D(\cdot, v), g_D(\cdot, w)) = g_D(v, w) \quad \text{for } v, w \in D. \quad (11.87)$$

In a strict sense the Dirichlet integrals in (11.84) and (11.87) exist only as improper integrals, which is reflected in the removal of small disks around the points  $\infty$  and  $w$  in (11.74) and (11.85), respectively. There exist techniques to overcome this specific problem, as for instance, the use of local smoothing techniques at the singularity of the Green function, which is demonstrated in detail in [15], Chapter 1, §5.

In the next step of the proof we assume that  $D \setminus V$  has a smooth boundary  $\partial(D \setminus V)$ . It follows then that the planar Lebesgue measure of  $\partial V$  is zero, i.e.,  $m(\partial V) = 0$ , and further that in (11.82) we have equality for all  $z \in V$ . By  $g$  we denote the Green potential in (11.81), i.e.,

$$g := \int g_D(\cdot, v) d\mu(v). \quad (11.88)$$

Because of  $m(\partial V) = 0$ , we have

$$D_{D \setminus V}(u) = D_D(u), \quad (11.89)$$

and with (11.81) and (11.88), we rewrite the Dirichlet integral in (11.89) as

$$D_{D \setminus V}(u) = D_D(h) + 2 D_D(h, g) + D_D(g). \quad (11.90)$$

Then we deduce with Fubini's Theorem from (11.84) that

$$D_D(h, g) = \int D_D(h, g_D(\cdot, v)) d\mu(v) = 0, \quad (11.91)$$

and analogously from (11.87) that

$$D_D(g) = \iint D_D(g_D(\cdot, v), g_D(\cdot, w)) d\mu(v) d\mu(w) = \iint g_D(v, w) d\mu(v) d\mu(w). \quad (11.92)$$

Putting (11.90), (11.91), and (11.92) together, we have proved identity (11.83) for the case that  $\partial(D \setminus V)$  is sufficiently smooth.

In the general situation, identity (11.83) follows, as in the proof of Lemma 37, by exhausting the open set  $D \setminus V$  by a sequence of nested open sets with sufficiently smooth boundaries.  $\square$

**11.4. Sequences of Compact Sets  $K_n$ .** Let  $K_n \subset \mathbb{C}$ ,  $n \in \mathbb{N}$ , be a sequence of compact sets of positive capacity. Because of the weak\*-compactness of the set of probability measures supported on a compact set, we know that any infinite subsequence  $N \subset \mathbb{N}$  contains an infinite subsequence, which we continue to denote by  $N$ , such that the equilibrium measures  $\omega_n = \omega_{K_n}$  of  $K_n$  converge weakly in  $\overline{\mathbb{C}}$ , i.e., there exists a probability measure  $\omega_0 = \omega_{0,N}$  in  $\overline{\mathbb{C}}$  with

$$\omega_n \xrightarrow{*} \omega_0 \quad \text{as } n \rightarrow \infty, n \in N. \quad (11.93)$$

If in addition to (11.93) also the limit

$$\lim_{n \rightarrow \infty, n \in N} \text{cap}(K_n) =: c_0 > 0 \quad (11.94)$$

exists and the inequality in (11.94) holds true, then it follows from the Lower Envelope Theorem 16 of potential theory that for the sequence of Green functions  $g_{\Omega_n}(\cdot, \infty)$ ,  $n \rightarrow \infty$ ,  $n \in N$ , which is associated with the compact sets  $K_n$  via the outer domains  $\Omega_n = \Omega_{K_n}$ , we have the asymptotic relation

$$\limsup_{n \rightarrow \infty, n \in N} g_{\Omega_n}(\cdot, \infty) \leq -p(\omega_0; \cdot) - \log \text{cap}(c_0) =: g_{0,N}, \quad (11.95)$$

and equality holds in (11.95) quasi everywhere in  $\mathbb{C}$ .

Like the measure  $\omega_0 = \omega_{0,N}$  in (11.93), so also the function  $g_0 = g_{0,N}$  in (11.95) depends on the subsequence  $N \subset \mathbb{N}$ .

For all infinite subsequences  $N \subset \mathbb{N}$ , for which the two limits (11.93) and (11.94) exist, the potential  $g_{0,N}$  in (11.95) is well-defined, and it is an immediate consequence of (11.43) and the inequality in (11.95) that

$$g_{0,N}(z) \geq 0 \quad \text{for all } z \in \mathbb{C}. \quad (11.96)$$

LEMMA 41. *Let  $E \subset \mathbb{C}$  be a compact set, and let  $N \subset \mathbb{N}$  be an infinite subsequence for which the two limits (11.93) and (11.94) exist. The inclusion  $E \subset K_n$  for all  $n \in N$  implies that*

$$g_{0,N}(z) = 0 \quad \text{for quasi every } z \in E. \quad (11.97)$$

PROOF. Without loss of generality we can assume that  $\text{cap}(E) > 0$ . From  $E \subset K_n$ ,  $n \in N$ , it follows that

$$g_{\Omega_n}(z, \infty) \leq g_{\Omega_E}(z, \infty) \quad \text{for } z \in \mathbb{C}, \quad \Omega_n = \Omega_{K_n}, \quad (11.98)$$

(cf. [26], Corollary 4.4.5), and consequently we have

$$\limsup_{n \rightarrow \infty, n \in N} g_{\Omega_n}(z, \infty) \leq g_{\Omega_E}(z, \infty) \quad \text{for } z \in \mathbb{C}. \quad (11.99)$$

From (11.43), we know that  $g_{\Omega_E}(z, \infty) = 0$  for quasi every  $z \in \widehat{E}$  ( $\widehat{E}$  denotes the polynomial-convex hull of  $E$ ), and from (11.95) and (11.99), we then conclude that  $g_{0,N}(z) = 0$  for quasi every  $z \in \widehat{E}$ , which proves (11.97).  $\square$

The next lemma has played a key role at several places in the proof of existence of an extremal domain in Subsection 9.2. The proof of the lemma relies heavily on Carathéodory's Kernel Convergence Theorem, which will be stated just after the next lemma.

LEMMA 42. *Let  $R \subset \overline{\mathbb{C}}$  be a ring domain with  $\infty \in R$ ,  $A_1, A_2 \subset \mathbb{C}$  the two components of  $\overline{\mathbb{C}} \setminus R$ , let further  $N \subset \mathbb{N}$  be an infinite subsequence for which the two limits (11.93) and (11.94) exist, and for which therefore the function  $g_{0,N}$  in (11.95) is well-defined, and let further  $0 < r < \infty$  be an appropriately chosen constant.*

*If for each  $n \in N$  there exists a continuum  $V_n \subset K_n \cap \{|z| \leq r\}$  that intersects the ring domain  $R$ , i.e., we have*

$$V_n \cap A_j \neq \emptyset \quad \text{for } j = 1, 2, \quad (11.100)$$

*then*

$$V_0 := \overline{\mathbb{C}} \setminus \Omega \left( \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m, n \in N} V_n} \right) \quad (11.101)$$

*is a continuum with  $V_0 \subset \{|z| \leq r\}$  that also intersects  $R$ , i.e., we have*

$$V_0 \cap A_j \neq \emptyset \quad \text{for } j = 1, 2, \quad (11.102)$$

*and further we have*

$$g_{0,N}(z) = 0 \quad \text{for all } z \in V_0 \quad (11.103)$$

*where  $g_{0,N}$  denotes the function defined in (11.95). By  $\Omega(\cdot)$  we denote the outer domain in (11.101).*

It has already been mentioned that the proof of Lemma 42 is based on Carathéodory's Kernel Convergence Theorem from geometric function theory, which establishes an equivalence between an analytic and a geometric description of the convergence of a sequences of conformal mapping functions (cf. [24], Chapter 1.4).

THEOREM 19 (Carathéodory's Kernel Convergence Theorem). *Let  $\{\varphi_n\}_{n \in \mathbb{N}}$  be a sequence of univalent functions defined in  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  with  $\varphi_n(\infty) = \infty$  and*

$$0 < m_0 \leq \varphi'_n(\infty) \leq M_0 < \infty \quad \text{for each } n \in \mathbb{N}. \quad (11.104)$$

*The sequence of functions  $\varphi_n$ ,  $n \in \mathbb{N}$ , convergence locally uniformly in  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  to an univalent function  $\varphi_0$  if, and only if, the domains*

$$D_N = \text{Ker} \left( \{\varphi_n(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})\}_{n \in N} \right) := \Omega \left( \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m, n \in N} \overline{\mathbb{C}} \setminus \varphi_n(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})} \right) \quad (11.105)$$

*are identical for all infinite subsequences  $N \subset \mathbb{N}$ . In (11.105), the outer domain is denoted by  $\Omega(\cdot)$ . The domain  $D_N$  associated with the sequence  $N \subset \mathbb{N}$  is called kernel of the sequence of domains  $\varphi_n(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ ,  $n \in N$ .*

*If the sequence  $\{\varphi_n\}_{n \in \mathbb{N}}$  converges locally uniformly in  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ , then the limit function*

$$\varphi_0 = \lim_{n \rightarrow \infty} \varphi_n \quad (11.106)$$

*is the Riemann mapping function of  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  onto the domain  $D_{\mathbb{N}} \subset \overline{\mathbb{C}}$  with  $\varphi_0(\infty) = \infty$  and  $m_0 \leq \varphi'_0(\infty) \leq M_0$ .*

REMARK 6. *The restrictions (11.104) placed on  $\varphi'_n(\infty)$  make sure that degenerated cases are excluded.*

PROOF OF LEMMA 42. Let  $\varphi_n$  be the Riemann mapping function from  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  onto  $\overline{\mathbb{C}} \setminus V_n$  for each  $n \in \mathbb{N}$  with

$$\varphi_n(\infty) = \infty \quad \text{and} \quad \varphi_n'(\infty) > 0. \quad (11.107)$$

It is immediate that

$$g_{\overline{\mathbb{C}} \setminus V_n}(z, \infty) = \log |\varphi_n^{-1}(z)| \quad \text{for } z \in \overline{\mathbb{C}} \setminus V_n, \quad (11.108)$$

and we have  $g_{\overline{\mathbb{C}} \setminus V_n}(z, \infty) = 0$  for all  $z \in V_n$  since continua are regular sets. From Lemma 32 it follows that

$$\text{cap}(V_n) = \varphi_n'(\infty). \quad (11.109)$$

From Lemma 20 and (11.109), we conclude that the assumptions  $V_n \subset \{|z| \leq r\}$  and  $V_n \cap A_j \neq \emptyset$  for  $j = 1, 2$  and  $n \in \mathbb{N}$  imply that

$$\text{dist}(A_1, A_2)/4 \leq \text{cap}(V_n) \leq r \quad \text{for all } n \in \mathbb{N}. \quad (11.110)$$

From (11.109) and (11.110), it then follows that the restrictions (11.104) in Theorem 19 are satisfied.

From (11.110) and the assumption  $V_n \subset \{|z| \leq r\}$  for all  $n \in \mathbb{N}$ , it further follows that there exists an infinite subsequence  $N \subset \mathbb{N}$  such that the two limits (11.93) and (11.94) exist, and therefore also the limit function  $g_{0,N}$  in (11.95) exists with  $K_n$  replaced by the sets  $V_n$ , and the outer domain  $\Omega_n$  by  $\overline{\mathbb{C}} \setminus V_n$  for each  $n \in N$ .

Because of  $V_n \subset \{|z| \leq r\}$  for  $n \in \mathbb{N}$ , we have a proper limit and locally uniform convergence in  $\{|z| > r\}$  in (11.95). With (11.108), this implies that the sequence  $\varphi_n$ ,  $n \in N$ , converges uniformly in a closed neighborhood of infinity. From the convergence together with the property that the  $\varphi_n$  are mapping functions into  $\overline{\mathbb{C}} \setminus V_n$  and the property (11.107), we deduce that  $\{\varphi_n, n \in N\}$  forms a normal family in  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ , and by Montel's Theorem together with the convergence in  $\{|z| > r\}$ , it then follows that

$$\lim_{n \rightarrow \infty, n \in N} \varphi_n(z) =: \varphi_{0,N}(z) \quad (11.111)$$

holds locally uniformly for  $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ .

From Carathéodory's Kernel Convergence Theorem, we then concluded that the limit function  $\varphi_0$  in (11.111) is the Riemann mapping function of  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  onto the domain  $D_N \subset \overline{\mathbb{C}}$  defined by (11.105) with the subsequence  $N$ , which has been used in (11.111), and from (11.105) and (11.111), we further know that

$$V_0 := \overline{\mathbb{C}} \setminus D_N \quad (11.112)$$

is a continuum. From (11.105) we learn that

$$V_0 = Pc\left(\bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m, n \in N} V_n}\right) = \overline{\mathbb{C}} \setminus \Omega \left( \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m, n \in N} V_n} \right) \quad (11.113)$$

with  $Pc(\cdot)$  denoting the polynomial-convex hull. For the two components  $A_1$  and  $A_2$  of  $\overline{\mathbb{C}} \setminus R$  we have

$$A_j \cap \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m, n \in N} V_n} = \bigcap_{m \in \mathbb{N}} \overline{(A_j \cap \bigcup_{n \geq m, n \in N} V_n)} = \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m, n \in N} (A_j \cap V_n)}, \quad (11.114)$$

$j = 1, 2$ . Since we have assumed that  $V_n \cap A_j \neq \emptyset$  for  $j = 1, 2$  and all  $n \in N$ , we conclude from (11.114) that

$$A_j \cap \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m, n \in N} V_n} \neq \emptyset \quad \text{for } j = 1, 2, \quad (11.115)$$

since the intersections are nested. From (11.115) and (11.113), we immediately get

$$A_j \cap V_0 \neq \emptyset \quad \text{for } j = 1, 2. \quad (11.116)$$

Since we have assumed  $V_n \subset K_n$ , we have

$$g_{\Omega_n}(z, \infty) \leq g_{\mathbb{C} \setminus V_n}(z, \infty) \quad \text{for } z \in \mathbb{C} \quad (11.117)$$

and all  $n \in N$  and  $\Omega_n := \Omega_{K_n}$  (cf. [26], Corollary 4.4.5). From the convergence (11.111) together with the identities in (11.108), we conclude that

$$\lim_{n \rightarrow \infty, n \in N} g_{\mathbb{C} \setminus V_n}(z, \infty) = g_{\mathbb{C} \setminus V_0}(z, \infty) \quad (11.118)$$

holds locally uniformly for  $z \in \mathbb{C}$ . From (11.117) together with limit relation (11.105) and (11.118), it then follows that

$$g_{0,N}(z) \leq g_{\mathbb{C} \setminus V_0}(z, \infty) \quad \text{for quasi every } z \in \mathbb{C}, \quad (11.119)$$

where  $g_{0,N}$  is the limit function in (11.118). Because of the first inequality in (11.110),  $V_0$  is of positive capacity, and therefore the equilibrium measure  $\omega_{V_n}$  of  $V_n$  is of finite energy. With the principle of domination in Theorem 17, it then follows that the inequality in (11.119) holds for all  $z \in \mathbb{C}$ . Since  $g_{\mathbb{C} \setminus V_0}(z, \infty) = 0$  for all  $z \in V_0$ , conclusion (11.103) of Lemma 42 follows from (11.119). The two other conclusions (11.101) and (11.102) are identical with (11.113) and (11.116), which completes the proof of the lemma.  $\square$

### 11.5. Critical Trajectories of Quadratic Differentials.

Let  $q$  be a function meromorphic in a domain  $D \subset \mathbb{C}$ . In (5.2), trajectories of the quadratic differential  $q(z)dz^2$  have been defined as smooth Jordan arcs  $\gamma$  with parametrization  $z : [0, 1] \rightarrow \overline{\mathbb{C}}$  that satisfy the relation

$$q(z(t))\dot{z}(t)^2 < 0 \quad \text{for } t \in (0, 1). \quad (11.120)$$

Like in Section 5.2, we use [40] and [10] as general reference for quadratic differentials.

Assertions about the global behavior of trajectories of a quadratic differential  $q(z)dz^2$  are difficult to obtain, but the situation is dramatically different with respect to their local behavior; it depends only on the local form of the function  $q$ , and is basically a consequence of the degree of its poles and zeros. Further, it is not difficult to see that the qualitative behavior of trajectories is invariant under conformal maps.

All zeros and simple poles of the function  $q$  are called *finite critical points* of the quadratic differential  $q(z)dz^2$ , and trajectories that end at zeros and poles are called *critical*. In the next lemma we assemble results about the local behavior of trajectories that have been used at several places of our analysis, further above. These results are not difficult to prove, and proofs can be found in [10], Chapter 8.2.

LEMMA 43. We consider a quadratic differential  $q(z)dz^2$ , and assume that  $q$  is meromorphic in a domain  $D \subset \mathbb{C}$ .

(i) If  $q$  is analytic in a neighborhood  $U$  of  $z_0 \in D$  and if  $q(z) \neq 0$  for all  $z \in U$ , then all trajectories of  $q(z)dz^2$  are laminar in  $U$ .

(ii) Let  $z_0 \in D$  be a finite critical point of the quadratic differential  $q(z)dz^2$ , i.e., at  $z_0$  the function  $q$  has the local behavior

$$q(z) = q_0(z - z_0)^l + O((z - z_0)^{l+1}) \quad \text{as } z \rightarrow z_0, \quad q_0 \neq 0, \quad (11.121)$$

and let further  $U$  be a neighborhood of  $z_0$  with  $q(z) \neq 0, \infty$  for all  $z \in U \setminus \{z_0\}$ , then  $l + 2$  trajectories of  $q(z)dz^2$  end at the point  $z_0$ , and they form a regular star at  $z_0$ , i.e., all angles between neighboring trajectories are equal to  $2\pi/(l + 2)$ . All other (non-critical) trajectories of  $q(z)dz^2$  are laminar in  $U$ .

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